Lucy and Lily: A Game of Geometry and Number Theory

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1. INTRODUCTION. The purpose of this article is to describe a computer game I created. I named the game "Lucy and Lily," after my two daughters. If you want to play this game, you can find it on my website www.math.umd.edu/∼res. Some browsers have trouble with the game, but some work just fine. Many people have expressed enthusiasm for the game, and their enthusiasm has encouraged me to write this article. At some point, Daniel Allcock and Brian Conrad worked out an informal but careful analysis of "Lucy and Lily." The several challenges I issue to the reader, during the course of the article, derive from facts that one or both of them established.

2. THE GAME IN GENERAL. Imagine that P is a convex polygon, placed at some initial position in the plane. Imagine also that P likes to wander away from the initial position when you are not looking. P moves from place to place by reflecting itself through its sides. Your job is to move P back to the origin, using the same kinds of reflection moves. If you are lucky, you might be able to figure out exactly which moves P made, and then simply undo them. Otherwise, you would have to find some other sequence of reflections to do the job.

Figure 1 shows two snapshots of a sample game, played with a certain irregular pentagon P. The number k indicates the position of P after the kth move. The left-hand side of Figure 1 records the first five moves and the right-hand side records the first nine moves. We have lightly shaded the last few positions on the right-hand side, so as to clarify the picture. Note that P is allowed to cross back over a path it has taken at an earlier time.

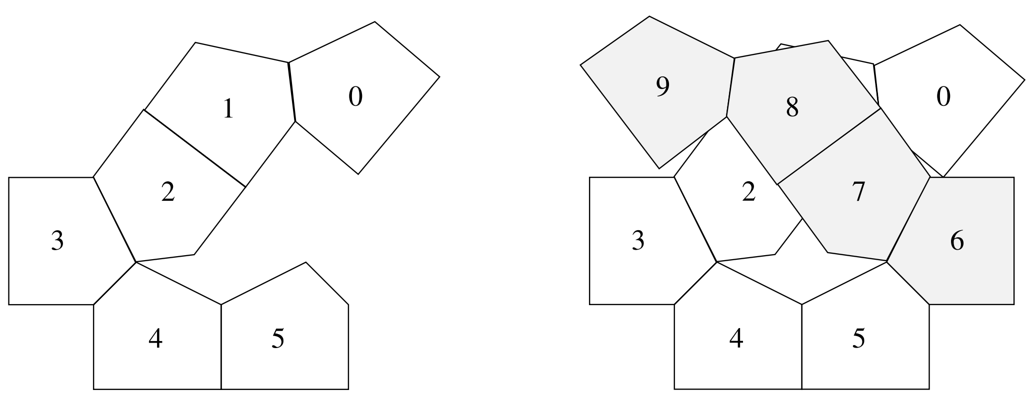

Figure 1.

3. SIMPLE CASES. The game is particularly simple when P is an equilateral triangle, a square, or a regular hexagon. In all these cases, we scale P so that the distance from the center of P to the center of any edge is $1/2$. We identify the plane with $\boldsymbol{C}$, the set of complex numbers, and we center P at 0. To pin down the initial position of P exactly, we rotate so that an edge of P is contained in the vertical line $\{x + iy \mid y = 1/2\}$. Compare Figures 2 and 3 below.

We introduce some notation to help with our analysis. Let $\mathbf{Z}$ denote the integers and let

$$\omega_n = \exp(2\pi i/n) = \cos(2\pi/n) + i \sin(2\pi/n).$$

Then ω_n is an nth root of unity. Let $\mathbf{Z}[\omega_n]$ denote the set of linear combinations of the form

$$\sum_{i=0}^{n-1} m_i \omega_n^i; \qquad m_0, \ldots, m_{n-1} \in \mathbf{Z}. \tag{1}$$

For instance, $\mathbf{Z}[\omega_4] = \{a + bi \mid a, b \in \mathbf{Z}\}$, since $\omega_4 = i$.

Figure 2 shows the case when P is the unit square. The shaded square represents the initial position. After the first move P is centered at one of the four points $\pm 1, \pm i$. In general, any sequence of moves centers P at a point of $\mathbf{Z}[\omega_4]$.

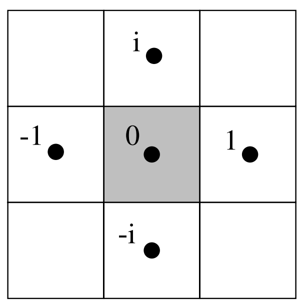

Figure 2.

From a geometric point of view, the plane is tiled by the usual grid of squares. Any sequence of moves lands P exactly on top of one of the tiles. The set of centers of the tiles is precisely $\mathbf{Z}[\omega_4]$, and one can move P so as to coincide with any of these tiles. Therefore, the set of possible centers of P is precisely $\mathbf{Z}[\omega_4]$. To return P to the origin from any given position, you simply move P from tile to tile, steadily decreasing the distance to the origin.

In case P is an equilateral triangle, any sequence of moves lands P on top of one of the tiles of the usual tiling of the plane by equilateral triangles.

In Figure 3 we have labelled some of the centers of these tiles. (We have set $\omega = \omega_3$ in the picture.) After the first move, P is centered at one of the three points 1, ω_3, or ω_3^2. After the second move, P is centered at a point of the form $\omega_3^i - \omega_3^j$, for i, j in $\{0, 1, 2\}$. After the third move, P is centered at a point of the form $\omega_3^i - \omega_3^j + \omega_3^k$. And so forth. In general, any sequence of moves places the center of P in $\mathbf{Z}[\omega_3]$. Looking at the way in which these sums are generated by the moves of P, we see that there is a restriction on the set of possible centers of P; namely, referring to (1), the sum $m_0 + m_1 + m_2$ is congruent to 0 mod 3 after an even number of moves and congruent to 1 mod 3 after an odd number of moves. The first challenge problem shows that this is the only restriction.

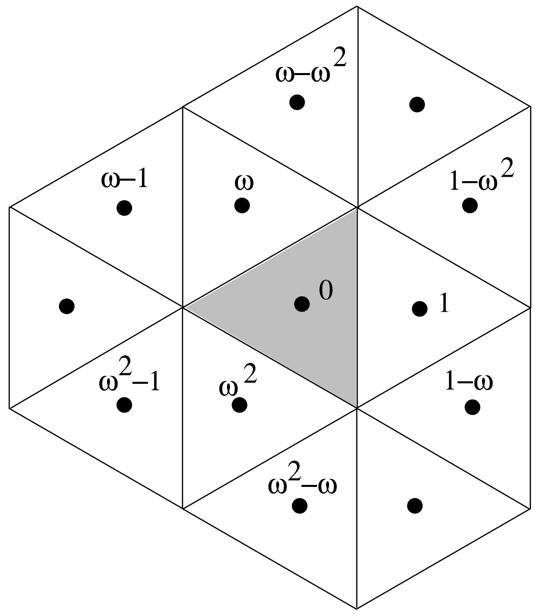

Figure 3.

Challenge 1: Suppose integers m_0, m_1 and m_2 are such that $m_0 + m_1 + m_2$ is congruent to 0 or 1 mod 3. Prove that one can move P so that P is centered at the point $m_0 + m_1\omega_3 + m_2\omega_3^2$. (Hint: You might want to use the relation that $1 + \omega_3 + \omega_3^2 = 0$.)

When P is a regular hexagon, any sequence of moves lands P on one of the tiles of the familiar tiling of the plane by regular hexagons. The set of centers of these tiles is exactly $\mathbf{Z}[\omega_6]$.

4. THE REGULAR PENTAGON. The fun starts when P is a regular pentagon, shown as the shaded part of Figure 4. We normalize P as in the previous section and

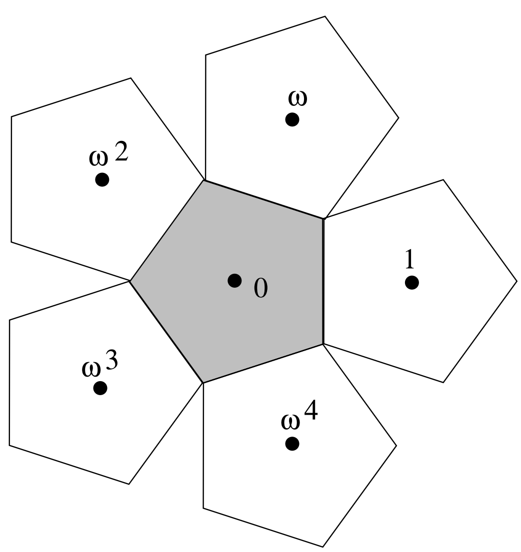

Figure 4.

we set $\omega = \omega_5$. Figure 4 shows the possible positions to which P can relocate in a single move.

One cannot tile the plane by regular pentagons and a little experience indicates that one can move P so as to be centered essentially anywhere in the plane.

Challenge 2: Prove that one can, by a sequence of reflections, move P so that its center is arbitrarily close to any preselected point in the plane. (You may want to contemplate Challenge 3 before trying this problem.)

In general, the center of P must always lie in $\mathbf{Z}[\omega]$. As in the case of the equilateral triangle, there is a restriction on the set of sums which can arise in our game.

Challenge 3: Prove that the set of possible centers of P is exactly the set of points in $\mathbf{Z}[\omega]$ for which $\sum_{i=0}^{4} m_i$ is congruent to 0 or 1 mod 5. As in the equilateral triangle case, the sum is congruent to 0 after an even number of moves and congruent to 1 after an odd number of moves.

Let us personalize the pentagon game a bit. We call the pentagon *Lucy* and color it as shown in Figure 6. Lucy wanders away from the origin (while you are sleeping), and your job is to get up shortly thereafter and return her to her initial position. You will discover that this task is extremely difficult (especially if you were hoping to sleep through the night!). But seriously, the *mathematical* problem is that Lucy can occupy a dense set of positions, so it is hard to tell, from sight, whether a given position is close or far, in terms of allowable moves, from the initial one.

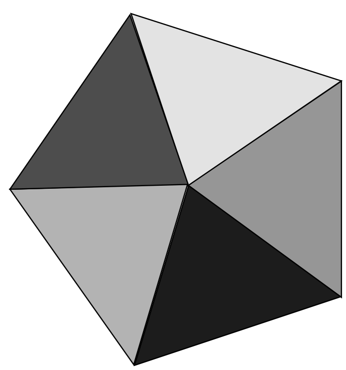

Figure 5.

Challenge 4: Prove that Lucy cannot be returned to the origin in such a way that the colors are in different positions from their initial ones.

5. A MAGIC TRICK. One can add a new twist. Suppose that there is an alternate universe which contains another pentagon, which we call *Lily*. Figure 6 shows the two planar universes in parallel. Lucy is shown on the left and Lily is shown on the right. Notice that the labelling on the right is different from the labelling on the left. The "distance around" between successive numbers on the right is twice the "distance around" between the corresponding numbers on the left.

Suppose the two sisters are in constant mental contact, and coordinate their moves. Every time Lucy reflects herself though an edge, Lily reflects herself through the same-colored (or, equivalently, same-numbered) edge, and *vice versa*. You, the player, are

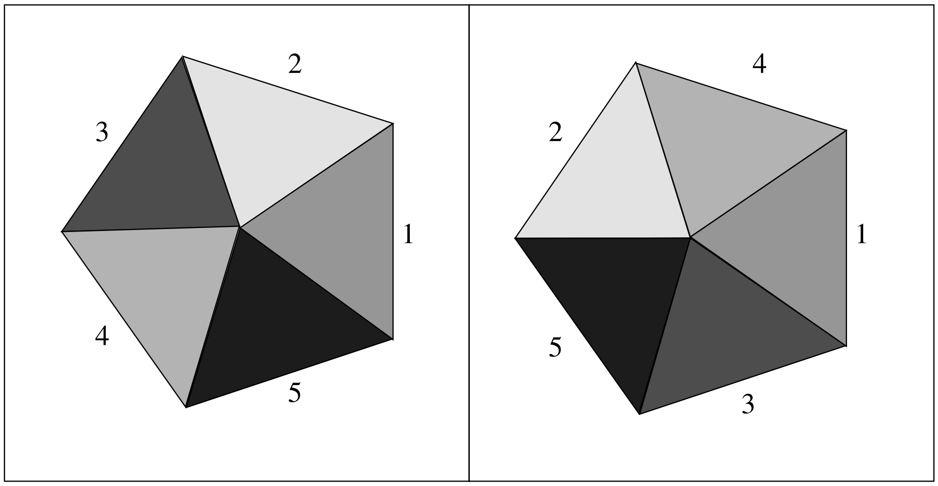

Figure 6.

allowed to see what is happening in both universes and can direct moves in either universe. However, a move in one universe forces the corresponding move in the other one.

Here is a strategy you can use to restore Lucy and Lily to their respective origins *simultaneously*.

- Move Lucy directly towards the origin until she touches the origin.
- Switch universes. Repeat Step 1 for Lily.
- Switch universes. Repeat Step 1 for Lucy.
- Switch universes. Repeat Step 1 for Lily.

And so forth. After finitely many steps you will notice that your job is done. If you want to see this strategy in action, you should go to my website and try it out. You will see that it works like magic.

6. BEHIND THE MAGIC TRICK. Why does the magic trick work? First of all, let's see how to move Lucy "directly towards the origin." If you move Lucy so as to cover a lot of ground efficiently, you do so by reflecting though edges that, in succession, are not adjacent to each other. If you move Lucy by reflecting though edges that, in succession, are adjacent to each other, then she covers ground inefficiently. The same remarks apply to Lily.

Here is the key observation. Adjacent edges on Lucy correspond to non-adjacent edges on Lily, and *vice versa*. Thus, if Lucy is moving efficiently, Lily is moving inefficiently. While Lucy strides towards her goal, Lily is forced by the rules to waddle around. Even if Lily waddles away from her goal, the amount of ground she loses is less than the amount of ground Lucy gains. All these remarks apply with the roles of the two pieces switched. From this analysis, one can see that two steps of our strategy will decrease the combined distance of the sisters to the origin.

Suppose that $R > 0$ is some fixed positive number. We claim that there are only finitely many possible positions for Lucy and Lily such that the combined distance of the sisters to the origin is less than R. Assuming this *finiteness claim*, we see that our strategy keeps decreasing the number of allowable positions for the two pieces until the only position left is the desired one.

We now prove our finiteness claim. Suppose that C_1 and C_2 are respectively the centers of Lucy and Lily at some stage of a game. We already know that C_1 and C_2 belong to $\mathbf{Z}[\omega]$. How are C_1 and C_2 related? Comparing the numbering of the edges in Figure 6 with the locations of powers of ω shown in Figure 4, one can see what happens after one move: $C_1 = \omega^j$ and $C_2 = \omega^{2j}$. One can prove by induction that $C_2 = \rho(C_1)$, where

$$\rho\left(\sum_{i=0}^{4} m_i \omega^i\right) = \sum_{i=0}^{4} m_i \omega^{2i}. \tag{2}$$

You might recognize ρ as the restriction to $\mathbf{Z}[\omega]$ of a generator of the Galois group of the cyclotomic field $\boldsymbol{Q}[\omega]$. (See [**3**, §4.11].) This fact is not actually needed in our proof, but it is certainly nice to know.

Define

$$C_k = \sum_{i=0}^{4} m_i \omega^{ki}; \qquad k = 1, 2, 3, 4.$$

As we already noted, if C_1 records Lucy's position at a certain point in the game, then C_2 records Lily's position at the same point in the game. The roles of C_3 and C_4 will become clear shortly. Note that ω^{2i} is always the complex conjugate of ω^{3i}. Hence, C_2 and C_3 are complex conjugates. That is, $C_2 = \overline{C}_3$. Similarly $C_1 = \overline{C}_4$.

The Cs are the roots of the degree 4 polynomial

$$Q(z) = \prod_{i=1}^{4}(z - C_i) = \sum_{j=0}^{4} A_j z^j.$$

If the combined distance from Lucy/Lily to the origin is less than R, then $|C_1| + |C_2| < R$. Since complex conjugation preserves absolute values we have

$$|C_j| < R \quad \text{for } j = 1, 2, 3, 4.$$

The As are *universal* polynomials in the Cs: i.e., the formulas that define the As in terms of the Cs do not depend on the Cs. The bounds on the Cs therefore give:

$$|A_j| < R'$$

for $j = 0, 1, 2, 3, 4$, where R' only depends on R. In the next paragraph, we will prove that $4A_j$ is an integer for each j. Assuming this for the moment, we observe that our last bound leaves only finitely many choices for the polynomial Q. Each choice has only finitely many roots, and each root corresponds to at most 5 positions each for Lucy and Lily. This establishes the finiteness claim.

To see that the $4A_j$ is an integer, let's write

$$A_j = \sum_{i=0}^{4} m_{ji} \omega^i; \qquad m_{ji} \in \mathbf{Z}. \tag{3}$$

Our polynomial Q has the property that

$$A_j = \overline{A}_j = \rho(A_j) = \rho(\overline{A}_j).$$

(This property of Q derives from the fact that ρ is a generator of the Galois group of $\boldsymbol{Q}[\omega]$, as we mentioned earlier.) In conjunction with (2), (3), and the fact that $\overline{\omega} = \omega^4$, this leads to

$$4A_j = A_j + \overline{A}_j + \rho(A_j) + \rho(\overline{A}_j) = 4m_{j0} + \sum_{i=1}^{4} m_{ji} \sum_{j=1}^{4} \omega^j = 4m_{j0} - \sum_{i=1}^{4} m_{ji} \in \boldsymbol{Z}.$$

One final remark: It is a special case of [**3**, Theorem 4.24] that

$$\boldsymbol{Z} = \boldsymbol{Q} \cap \boldsymbol{Z}[\omega].$$

Our proof shows that A_j belongs to $\boldsymbol{Q}$, and by construction A_j lies in $\boldsymbol{Z}[\omega]$. Hence, we have the stronger result that A_j itself is an integer.

7. FOUR DIMENSIONAL INTERPRETATION. Let ρ be the map described by (2). We can consider the "graph" of ρ. That is, we consider the set

$$\Gamma = \{(x, \rho(x)) \mid x \in \boldsymbol{Z}[\omega]\} \subset \boldsymbol{C}^2.$$

Our proof of the finiteness claim shows that Γ is a discrete set of points in $\boldsymbol{C}^2$. The fact that $\boldsymbol{Z}[\omega]$ is an abelian group translates into the fact that adding two points of Γ always results in another point of Γ. This is only possible if Γ is a grid of points in $\boldsymbol{C}^2$. It is not hard to check that Γ is not contained in a lower dimensional subspace of $\boldsymbol{C}^2$. In other words, Γ must be a 4-dimensional grid—i.e., a *lattice*. This phenomenon is a special case of a general result about rings of algebraic integers. See, for instance, [**2**, p. 99].

To get a clearer picture of our magic trick, we should really think of Lucy and Lily as being 2-dimensional shadows of a 4-dimensional piece X=(Lucy,Lily). As X roams around a 4-dimensional grid we see two different 2-dimensional shadows of her motion, projected into alternate universes. The player, who has access to both universes, unconsciously uses the 4-dimensional space as a guide!

8. VARIATIONS ON A THEME. There is a lower dimensional variant of our game that shares many of its main features. Suppose a red dot is placed at the origin on the number line. When you are not looking, the dot moves around, repeatedly jumping to the left or right by either $\sqrt{2}+1$ units or $\sqrt{2}-1$ units. Your job is to return the dot to the origin using the same kinds of jumps. The dot can land at any point of the form $a + b\sqrt{2}$, where a and b are integers and $a + b$ is even. The density of this set of numbers makes our lower dimensional game difficult in the same way that the pentagon game, played with one pentagon, is difficult.

One can perform the same kind of magic trick as above by placing a blue dot on a second number line, and linking the two dots together as follows: if the red dot jumps $\sqrt{2}+1$ units to the left (respectively, right), the blue dot jumps $\sqrt{2}-1$ units to the right (respectively, left); if the red dot jumps $\sqrt{2}-1$ units to the left (respectively, right), the blue dot jumps $\sqrt{2}+1$ units to the right (respectively, left). In general, if the red dot is at the point $a + b\sqrt{2}$ the blue dot is at the point $\rho(a + b\sqrt{2}) = a - b\sqrt{2}$. The map ρ is the generator of the Galois group of $\boldsymbol{Q}[\sqrt{2}]$. The "graph"

$$\Gamma = \{\rho(x) \mid x \in \boldsymbol{Z}[\sqrt{2}]\}$$

is a grid of points in the plane. The vectors $(1, 1)$ and $(1+\sqrt{2}, 1-\sqrt{2})$ generate Γ.

You can easily return the *pair* of dots to the origin by moving the pair from grid point to grid point.

Going to higher dimensions rather than lower dimensions, we can set up a similar magic trick for regular n-gons. Rather than using two alternate universes, one uses $\phi(n)/2$ of them. Here ϕ is the *Euler phi function*, which counts the number of residue classes in $\mathbf{Z}/n$ that are relatively prime to n. For instance, $\phi(5) = 4$ and $\phi(7) = 6$. In the generalized version, you look at $\phi(n)/2$ shadows of a piece wandering around on a $\phi(n)$-dimensional grid. The n-gons in these alternate universes are carefully colored, so as to take advantage of the multiplicative structure of $\mathbf{Z}/n$.

I say that one can *set up* the magic trick rather than *perform* it, because I don't actually know an automatic winning strategy for a general regular n-gon. Some of our analysis for the pentagon game does not quite work in the general setting. If you want to try your hand at the more general game, and formulate your own strategy, you should check out "Lucy and Lily II," which is right next to "Lucy and Lily" on my website. "Lucy and Lily II" implements the cases $n = 5, \ldots, 13$. (One difference in the sequel is that the "alternate universes" are superimposed onto each other, so that all the action takes place in one "screen.")

Here are a few connections between "Lucy and Lily" and other topics:

- There is a striking parallel between our "alternate universe trick" and Jim Propp's use of the cities Hoboken and Manhattan in his analysis of skew tetromino tilings [**5**].
- "Lucy and Lily" can be modified to draw quasicperiodic tilings. The rough idea is to confine Lucy to stay near the origin and then to consider the set of all possible centers of Lily. These centers will be the vertices of what essentially is a Penrose tiling. See [**1**] for constructions related to this.
- The ideas behind "Lucy and Lily" are baby versions of the ideas which go into the theory of arithmetic lattices. The easiest cases of this vast theory are the arithmetic surface groups. See [**4**] for an elementary description.

I'll leave other variations on the theme to your imagination. Go to the website and check out the games!

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REFERENCES

1. Veit Elser, *Crystallography and Riemann Surfaces*, Discrete and Computational Geometry (2001) **25**:3, 445–476.
2. Z. I. Borevich and I. R. Shafarevich, *Number theory*, Academic Press, 1966.
3. Nathan Jacobson, *Basic Algebra I*, W.H. Freeman and Company, 1974.
4. Svetlana Katok, *Fuchsian Groups*, Chicago Lectures in Mathematics, University of Chicago Press, 1992.
5. James Propp, *A pedestrian approach to a method of conway, or, a tale of two cities*, Mathematics Magazine, December 1997.

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