The Optimal Paper Moebius Band

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Abstract

We prove that a smooth embedded paper Moebius band must have aspect ratio greater than $\sqrt{3}$. We also prove that any sequence of smooth embedded paper Moebius bands whose aspect ratio converges to $\sqrt{3}$ must converge, up to isometry, to the triangular Moebius band. These results answer the mimimum aspect ratio question discussed by W. Wunderlich in 1962 and prove the more specific conjecture of B. Halpern and C. Weaver from 1977.

1 Introduction

1.1 The Triangular Moebius Band

To make a paper Moebius band you give a $1 \times \lambda$ strip of paper an odd number of twists and then join the ends together. For long strips this is easy and for short strips it is difficult or impossible. Let me first discuss a beautiful example known as the *triangular Moebius band*. Figure 1a shows the triangular Moebius band. It is based on a $1 \times \sqrt{3}$ strip.

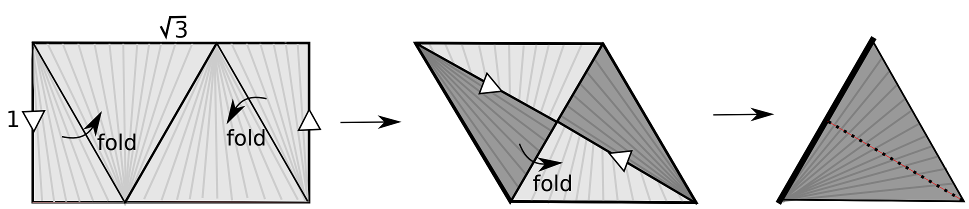

Figure 1a: The triangular Moebius band

The strip in Figure 1a is lightly shaded on one side and darkly shaded on the other. First fold the flaps in to make a rhombus, then fold the rhombus in half like a wallet. This folding brings the two ends together with a twist.

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The dotted segment indicates where the ends are joined. The bold segment indicates the "wallet fold". The dotted and bold segments together make a pattern like a T. The pinstriping exhibits the strip as a union of line segments, disjoint except at the endpoints, which stay straight during the folding.

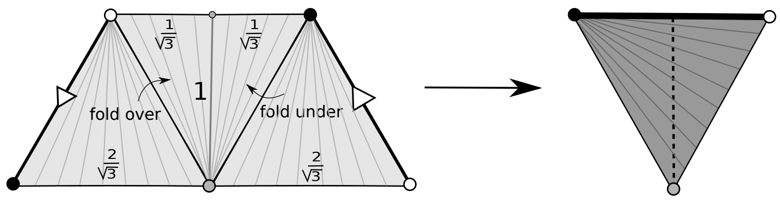

Figure 1b: The triangular Moebius band: another view

Figure 1b shows another view. Here we start with a symmetric trapezoid rather than a rectangle, but we get the same object when we fold and join the sides together. The bold edge indicates where the sides are joined. The dotted and bold segments again make a "T-pattern".

The triangular Moebius band goes back at least to the 1930 paper [**Sa**] of M. Sadowsky. Technically, it does not quite fit the definition of a (smooth, embedded) paper Moebius band that we give below, but it is the limit of such.

1.2 The Minimum Aspect Ratio Question

The triangular Moebius band looks like an extremely efficient construction. Can we do better in terms of making λ smaller? To answer this question in a meaninful way, we first need a formal definition.

Definition: A *smooth paper Moebius band of aspect ratio* λ is an infinitely differentiable isometric mapping $I : M_\lambda \to \boldsymbol{R}^3$, where M_λ is the flat Mobius band obtained by identifying the length-1 sides of a $1 \times \lambda$ rectangle. That is:

$$M_\lambda = ([0,\lambda] \times [0,1])/\sim, \qquad (0,y) \sim (\lambda, 1-y). \tag{1}$$

An *isometric mapping* is a map which preserves arc-lengths. The map is an *embedding* if it is injective, and an *immersion* in general. Let $\Omega = I(M_\lambda)$. We often write $I : M_\lambda \to \Omega$. We call Ω *embedded* when I is an embedding.

Remark: The smooth formalism rules out examples which render the main question meaningless. For instance, you could fold any rectangle (e.g. a square) like an accordion into a thin strip, twist, then tape. This origami monster is not the limit of smooth embedded paper Moebius bands. We give an alternative definition of a paper Moebius band a the top of §2.1 which is sufficient for our purposes and avoids smoothness.

The early papers of M. Sadowsky [**Sa**] and W. Wunderlich [**W**] treat both the existence and differential geometry of paper Moebius bands. (See [**HF**] and [**T**] respectively for modern English translations.) The paper [**CF**] gives a modern differential geometric framework for developable surfaces like Ω. The papers [**CK**], [**KU**], [**RR**] and [**Sab**] are all studies of the differential geometry of paper Moebius bands. I learned about paper Moebius bands from the great expository article [**FT**, Chapter 14] by Dmitry Fuchs and Sergei Tabachnikov.

W. Wunderlich discusses the minimum aspect ratio question, without an explicit guess, in the introduction of his 1962 paper [**W**]. In their 1977 paper [**HW**], Ben Halpern and Charles Weaver study the minimum aspect ratio question in detail. They prove two things.

- For smooth immersed paper Moebius bands one has $\lambda > \pi/2$. Moreover, for any $\epsilon > 0$ one can find an immersed example with $\lambda = \pi/2 + \epsilon$.
- There exists some $\epsilon_0 > 0$ such that $\lambda > \pi/2 + \epsilon_0$ for a smooth embedded paper Moebius band. This ϵ_0 is not an explicit constant.

On the last line of [**HW**], Halpern and Weaver conjecture that $\lambda > \sqrt{3}$ for a smooth embedded paper Moebius band.

1.3 Results

In this paper I will resolve the Halpern-Weaver conjecture.

Theorem 1.1 (Main) *A smooth embedded paper Moebius band has aspect ratio greater than* $\sqrt{3}$.

I will also prove that the triangular Moebius band is truly the optimal paper Moebius band – at least when it comes to minimizing the aspect ratio.

Theorem 1.2 (Triangular Limit) *Let* $\{I_n : M_{\lambda_n} \to \Omega_n\}$ *be a sequence of smooth embedded paper Moebius bands with* $\lambda_n \to \sqrt{3}$. *Then, up to isometry,* I_n *converges uniformly to the map giving the triangular Moebius band.*

Outline of the Proofs: Let $I : M_\lambda \to \Omega$ be a smooth embedded paper Moebius band. A *bend* is line segment $B' \subset \Omega$ which cuts across Ω and has its endpoints in the boundary. We call the pre-image $B = I^{-1}(B') \subset M_\lambda$ a *pre-bend*. Since I is arc-length preserving and B' is an arc-length minimizing path between its endpoints, B must also be an arc-length minimizing path between its endpoints. Hence B is also a line segment.

It is a classic fact that Ω has a continuous foliation β by bends. See §2.1. The bends in β vary continuously and are pairwise disjoint. The corresponding foliation of M_λ by pre-bends is like the pinstriping in Figures 1a and 1b except that the pre-bends are disjoint even at the endpoints.

We say that a *T-pattern* on Ω is a pair of bends which lie in perpendicular intersecting lines. Look again at the right sides of Figures 1a and 1b. We call the T-pattern *embedded* if the two bends are disjoint. In §2.2 we prove

Lemma 1.3 (T) *A smooth embedded paper Moebius band has an embedded T-pattern.*

Here is the idea. The space of pairs of unequal bends in β has a 2-point compactification which makes it into the 2-sphere, S^2. We define a pair of odd functions on S^2 which detect a T-pattern when they have a common zero. We apply the Borsuk-Ulam Theorem to get a common zero.

In §2.3 we prove

Lemma 1.4 (G) *A smooth embedded paper Moebius band with an embedded T-pattern has aspect ratio greater than $\sqrt{3}$.*

Here is the idea. We choose an embedded T-pattern on Ω and then cut M_λ open along one of the corresponding pre-bends. The result is a bilaterally symmetric trapezoid τ. See Figure 2 below. We then solve an optimization problem which involves mapping τ into $\boldsymbol{R}^3$ with constraints coming from the geometry of trapezoids and T-patterns.

The Main Theorem is an immediate consequence of Lemma T and Lemma G. In §2.4 we prove the Triangular Limit Theorem by examining what the proof of Lemma G says about a minimizing sequence of examples.

Remarks: (1) If an embedded paper Moebius band has a long enough strip that is contained in a single plane, it also has a non-embedded T-pattern. That is why we take special care to speak of embedded T-patterns.
(2) The interested reader would be able to tweak our proof of Lemma T to show that a smooth immersed Moebius band has a T-pattern, though not necessarily an embedded one. We do not want to fool around with this.
(3) Likewise, the interested reader would be able to tweak our proof of Lemma G to prove that an immersed paper Moebius band with an embedded T-pattern has aspect ratio greater than $\sqrt{3}$.
(4) The ideas above are an outgrowth of my earlier paper [**S1**]. In [**S1**] I prove a version of Lemma T in a complicated way and with the side hypothesis that $\lambda < 7\pi/12$. I then (correctly) deduce that $\lambda \geq \phi = (1+\sqrt{5})/2$. When I try to further improve this easy bound, I make an idiotic mistake: I claim that when you cut open M_λ along a pre-bend you get a parallelogram. This mistake invalidates my final bound, a weird and forgettable algebraic number in $(\phi, \sqrt{3})$. This paper supersedes [**S1**] and is independent from it.
(5) My informal notes [**S2**] give a slower and more elementary account of my proofs. I designed [**S2**] for college students and advanced high school students who want to learn the arguments.

1.4 Additional Material

The proofs are done after §2, but I include some more material in §3.

In §3.1 I elaborate on some aspects of the proofs given in §2.

In §3.2 I discuss an alternate framework for Lemma T.

In §3.3 I discuss some topics adjacent to paper Moebius bands. Let me also say a few things here. The paper [**CKS**] and [**DDS**] consider the related question of tying a piece of rope into a knot using as little rope as possible. The papers [**D**] and [**DL**] consider folded ribbon knots. [**DL**, Corollary 25] is in some sense a special case of our two results, and [**DL**, Conjecture 26] is a variant of the Halpern-Weaver Conjecture in the category of folded ribbon knots. Our Main Theorem incidentally resolves this folded ribbon knot conjecture. Some authors have considered "optimal Moebius bands" from other perspectives, either algebraic [**Sz**] or physical [**MK**], [**SH**].

In §3.4 I discuss some new results [**BrS**], [**H**], [**S3**] about multi-twist Moebius bands and cylinders which followed after the writing of this paper.

1.5 Acknowledgements

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2 Proofs of the Results

2.1 Existence of a Bend Foliation

One might instead define an embedded paper Moebius band to be an injective arc-length preserving map $I : M_\lambda \to \Omega \subset \boldsymbol{R}^3$ such that Ω has a continuous foliation by bends. This is what our proofs below really use. The reader who prefers this alternate definition can skip this section. In this section we deduce the existence of the bend foliation from the definition in the introduction.

Proposition 2.1 *Ω has a continuous foliation by bends.*

Proof: The *Gauss map* η, well-defined locally, assigns to each $p \in \Omega$ a unit vector η_p normal to Ω at p. Since Ω is smooth even at the boundary $\partial\Omega$, both η and the mean curvature make sense even in $\partial\Omega$.

Let $U \subset \Omega$ be the set of points with nonzero mean curvature. By either [**HN**, §3, Lemma 2] or [**CL**, p. 314, Lemma 2] each $p \in U$ lies in a unique bend $\gamma \subset U$. (See [**Mas**] or [**S4**] for simpler proofs; also see §3.1.) Hence U has a partition into bends. The disjointness of these bends implies their continuity.

Let τ be the closure of a component of $\Omega - U$. Note that τ lies in a single plane because η is constant on τ. Either τ is a single bend, the limit of bends converging to it on either side, or τ is a trapezoid: two opposite sides τ_1 and τ_2 are bends and the other sides lie in $\partial\Omega$. We foliate τ by bends, interpolating between τ_1 and τ_2. Doing this for all τ, we get our foliation of Ω. ♠

2.2 Proof of Lemma T

Definitions: Let $I : M_\lambda \to \Omega$ be a smooth embedded paper Moebius band. We choose a continuous foliation β of Ω by bends, as guaranteed by Proposition 2.1. The preimage $I^{-1}(\beta)$ is a continuous foliation of M_λ by pre-bends.

Each bend u has exactly 2 unit vectors $\pm\overrightarrow{u}$ parallel to it. We call either one an *orientation* of u. The *centerline* of M_λ is the circle $([0,\lambda] \times \{1/2\})/\sim$. The *centerline of* Ω is the image of the centerline of M_λ under the map I.

Intersection with the Centerline: Here is a proof that a pre-bend u intersects the centerline of M_λ exactly once. Let $\ell(\cdot)$ denote length. If we have $\ell(u) < \sqrt{1+\lambda^2}$ we can move u by an isometry so that it misses the vertical sides of $[0,\lambda] \times [0,1]$. But then u clearly intersects the centerline exactly once. So, if u intersects the centerline more than once, we have $\ell(u) \geq \sqrt{1+\lambda^2} > \lambda$. But $\partial\Omega = I(\partial M_\lambda)$ is a loop that contains the endpoints of the bend $u' = I(u)$. Hence $2\lambda = \ell(\partial\Omega) \geq 2\ell(u') = 2\ell(u) > 2\lambda$, a contradiction.

The Circle of Bends: Since I is an embedding, our intersection result above implies that each bend of Ω intersects the centerline of Ω exactly once. We associate to each bend of β the point where it intersects the centerline, which we identify with $\boldsymbol{R}/2\pi$. Thus we parametrize the bends of β by $\boldsymbol{R}/2\pi$.

The Cylinder and the Sphere: Let Υ be the topological cylinder of pairs $(x_0, x_1) \in (\boldsymbol{R}/2\pi)^2$ with $x_0 \neq x_1$. A point $(x_0, x_1) \in \Upsilon$ corresponds to a pair (u_0, u_1) of unequal bends. We let $\overline{\Upsilon}$ be the compactification of Υ obtained by adding 2 points: ∂_+ (respectively ∂_-) is the limit of pairs (x_0, x_1) where x_1 is just ahead (respectively just behind) x_0 in the cyclic order on $\boldsymbol{R}/2\pi$. The space $\overline{\Upsilon}$ is homeomorphic to S^2, the 2-sphere. (See §3.1 for an explicit homeomorphism.) The map $\Sigma(x_0, x_1) = (x_1, x_0)$ extends to a continuous involution of S^2 that swaps the two points ∂_+ and ∂_-.

Propagating the Orientations: Let $(x_0,x_1)\in\Upsilon$. There is a unique path $t\to x_t$ in $\boldsymbol{R}/2\pi$ which joins x_0 to x_1, moves at constant speed, locally increases in the cyclic order on $\boldsymbol{R}/2\pi$, and has length less than 2π. This path has length near 0 (respectively near 2π) when (x_0,x_1) is near ∂_+ (respectively ∂_-). Let u_t be the bend associated to x_t. We write $\overrightarrow{u}_0 \rightsquigarrow \overrightarrow{u}_1$ when there is a continuous orientation of the bends $\{u_t\}$ that restricts to $\overrightarrow{u}_0$ and $\overrightarrow{u}_1$. Note that $-\overrightarrow{u}_0 \rightsquigarrow -\overrightarrow{u}_1$ and, since Ω is a Moebius band, $\overrightarrow{u}_1 \rightsquigarrow -\overrightarrow{u}_0$. Also $\overrightarrow{u}_1$ converges to $\pm\overrightarrow{u}_0$ when (x_0,x_1) converges to $\partial_\pm$.

The Map: Let m_j be the midpoint of u_j. Using the dot product $(\cdot)$ and the cross product $(\times)$ define $F=(g,h):\Upsilon\to\boldsymbol{R}^2$, where

$$g(x_0,x_1)=\overrightarrow{u}_0\cdot\overrightarrow{u}_1, \qquad h(x_0,x_1)=(m_0-m_1)\cdot(\overrightarrow{u}_0\times\overrightarrow{u}_1). \qquad (2)$$

Here $\overrightarrow{u}_0 \rightsquigarrow \overrightarrow{u}_1$. Our definition is independent of the chosen orientation since $-\overrightarrow{u}_0 \rightsquigarrow -\overrightarrow{u}_1$. Also, F extends continuously S^2 with $F(\partial_\pm)=(\pm1,0)$. Since $\overrightarrow{u}_1 \rightsquigarrow -\overrightarrow{u}_0$ we have $g(x_1,x_0)=-g(x_0,x_1)$ and

$$h(x_1,x_0)=(m_1-m_0)\cdot(\overrightarrow{u}_1\times(-\overrightarrow{u}_0))=(m_1-m_0)\cdot(\overrightarrow{u}_0\times\overrightarrow{u}_1)=-h(x_0,x_1).$$

In short, $F\circ\Sigma=-F$.

The Common Zero: We have $(0,0)\in F(\Upsilon)$ by the Borsuk-Ulam Theorem. Here is a self-contained proof. Suppose not. If γ is a continuous path in S^2 which goes from ∂_+ to ∂_-, then $F(\gamma)$ goes from $(1,0)$ to $(-1,0)$, misses $(0,0)$, and winds some half integer $w(\gamma)$ times around the origin. All choices of γ are homotopic to each other relative to $\partial_\pm$, so $w(\gamma)$ is independent of γ. But consider $\gamma'=\Sigma(\gamma)$, re-oriented so that it goes from ∂_+ to ∂_-. Since $F\circ\Sigma=-F$ the image $F(\gamma')$ is obtained by rotating $F(\gamma)$ by 180 degrees about $(0,0)$ then re-orienting it so that it goes from $(1,0)$ to $(-1,0)$. But then $w(\gamma')=-w(\gamma)$, a contradiction.

Endgame: Let (u_0,u_1) be the bends corresponding to $(x_0,x_1)\in F^{-1}(0,0)$. First, u_0 and u_1 are disjoint because they belong to the same foliation. Second, $\overrightarrow{u}_0$ and $\overrightarrow{u}_1$ are orthogonal because $g(x_0,x_1)=0$. Third, $\overrightarrow{u}_0$ and $\overrightarrow{u}_1$ and m_0-m_1 are all orthogonal to $\overrightarrow{u}_0\times\overrightarrow{u}_1$ because $h(x_0,x_1)=0$, and this easily implies that u_0,u_1 are coplanar. Hence (u_0,u_1) is an embedded T-pattern. This proves Lemma T.

2.3 Proof of Lemma G

Let ℓ denote arc length. Let $\bigtriangledown$ be a triangle with horizontal base. Let $\vee$ be the union of the two non-horizontal sides of $\bigtriangledown$.

Lemma 2.2 *If* $\bigtriangledown$ *has base* $\sqrt{1+t^2}$ *and height* $h \geq 1$ *then* $\ell(\vee) \geq \sqrt{5+t^2}$, *with equality iff* $\bigtriangledown$ *is isosceles and* $h = 1$.

Proof: Let p_1, p_2, q be the vertices of $\bigtriangledown$, with p_1, p_2 lying on the base. Let p_2' be the reflection of p_2 in the horizontal line through q. Note that $\bigtriangledown$ is isosceles iff p_1, q, p_2' are collinear. By symmetry, the triangle inequality, and the Pythagorean Theorem,

$$\ell(\vee) = \|p_1 - q\| + \|q - p_2'\| \geq \|p_1 - p_2'\| = \sqrt{1 + t^2 + 4h^2} \geq \sqrt{5 + t^2}.$$

We get equality if and only if p_1, q, p_2' are collinear and $h = 1$. ♠

Let $I : M_\lambda \to \Omega$ be a smooth embedded paper Moebius band with an embedded T-pattern. Let $S' = I(S)$ for any $S \subset M_\lambda$. We have $\ell(\gamma) = \ell(\gamma')$ for any curve $\gamma \subset M_\lambda$. We rotate Ω so that one of the bends of the T-pattern, T', lies in X-axis and the other bend, B', lies in the negative ray of the Y-axis. Next, we let $B = I^{-1}(B')$ and $T = I^{-1}(T')$ be the corresponding pre-bends.

We cut M_λ open along T to get a bilaterally symmetric trapezoid τ. We normalize τ so that the parallel sides are horizontal. Reflecting τ in the coordinate axes if needed, we arrange that u, v, w, x are mapped to Ω as in Figure 2. Compare Figure 1b. The quantities t and b (which are both positive in the case depicted) respectively denote the horizontal displacements of T and B.

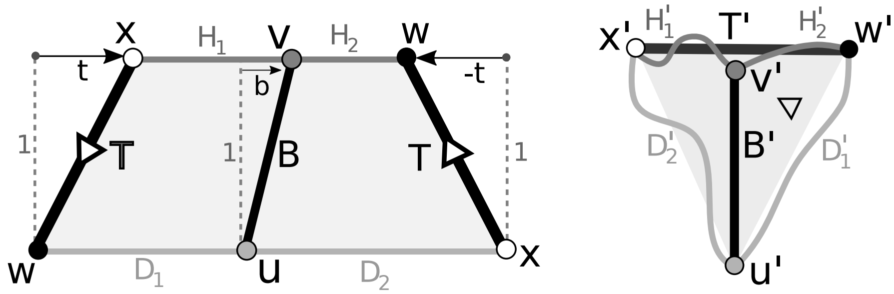

Figure 2: The trapezoid τ (left) and the T-pattern (right).

With Lemma 2.2 in mind, note that the shaded triangle $\bigtriangledown$ has

$$\text{base} = \ell(T') = \ell(T) = \sqrt{1+t^2}, \qquad \text{height} > \ell(B') = \ell(B) = \sqrt{1+b^2} \geq 1.$$

Let $H = H_1 \cup H_2$ and $D = D_1 \cup D_2$. We have $H', D' \subset \boldsymbol{R}^3$, so you should imagine you are floating in space above Figure 2 and looking down. Now H' connects the endpoints of T'. Also, D' connects the endpoints of T' and contains u. From this structure, and from Figure 2 (left), we have

$$\begin{aligned} \ell(H)+\ell(D) &= 2\lambda. \\ \ell(D)-2t &= \ell(H). \\ \sqrt{1+t^2} = \ell(T') \leq \ell(H') &= \ell(H). \\ \sqrt{5+t^2} <^{*} \ell(\vee) \leq \ell(D') &= \ell(D). \end{aligned} \tag{3}$$

The starred inequality is Lemma 2.2. Equation 3 give us

$$\begin{aligned} \delta(t) := \sqrt{1+t^2}+\sqrt{5+t^2} &< \ell(H)+\ell(D) = 2\lambda. \\ \nu(t) := 2\sqrt{5+t^2}-2t < 2\ell(D)-2t &= \ell(D)+\ell(H) = 2\lambda. \end{aligned} \tag{4}$$

Hence $2\lambda > \max(\delta(t), \nu(t))$.

Let $t_0 = 1/\sqrt{3}$. We have $\delta(t_0) = \nu(t_0) = 2\sqrt{3}$. Note the following.

- δ is increasing on $(0, \infty)$. Hence $\delta(t) > 2\sqrt{3}$ if $t > t_0$.
- ν is decreasing on $\boldsymbol{R}$. Hence $\nu(t) > 2\sqrt{3}$ if $t < t_0$.

Hence $\lambda > \sqrt{3}$. This proves Lemma G and, consequently, the Main Theorem.

2.4 Proof of the Triangular Limit Theorem

Figure 3 shows what Figure 2 looks like with respect to the triangular Moebius band and its T-pattern in Figure 1b.

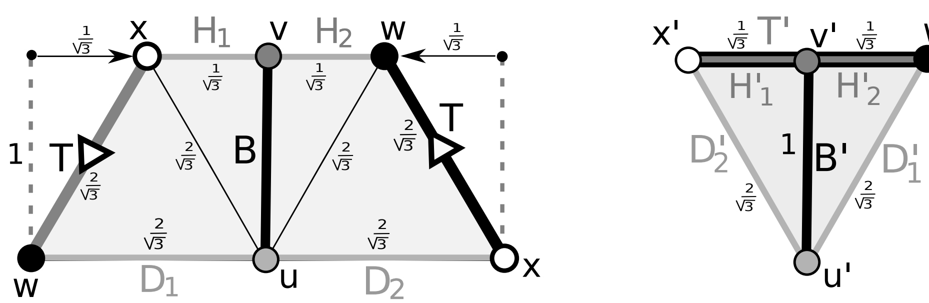

Figure 3: Figure 2 for the triangular Moebius band.

Let $\{I_n : M_{\lambda_n} \to \Omega_n\}$ be as in the Triangular Limit Theorem. We run the construction of Lemma G for each n and analyze what happens as $n \to \infty$. We use the same notation as in the proof of Lemma G, except that we add subscripts to indicate the dependence on n.

The Range: Since $\max(\delta(t_n), \nu(t_n)) \to 2\sqrt{3}$ we have $t_n \to 1/\sqrt{3}$. Hence $\ell(T'_n) \to 2/\sqrt{3}$. Since $\lambda_n \to \sqrt{3}$ and $\ell(\triangledown_n) \leq 2\lambda_n$, we have

$$\limsup \ell(\triangledown_n) \leq 2\sqrt{3}. \tag{5}$$

The base of $\triangledown_n$ converges to $2/\sqrt{3}$ and the height is always greater than 1. Lemma 2.2 combines with Equation 5 to show that $\triangledown_n$ converges (up to isometries) to an isosceles triangle of base $2/\sqrt{3}$ and height 1, namely the shaded equilateral triangle $\triangledown$ shown in Figure 3 (right). We normalize by isometries so that we get actual convergence.

The Domain: Since $\ell(\partial\Omega_n) - \ell(\triangledown_n) \to 0$ and also v'_n converges to the midpoint of T'_n, all the slack goes out of Equation 3, and

$$\lim \ell(H'_{n,1}) = \lim \ell(H'_{n,2}) = 1/\sqrt{3}, \quad \lim \ell(D'_{n,1}) = \lim \ell(D'_{n,2}) = 2/\sqrt{3}. \qquad (6)$$

Since $\ell(H_{n,1}) = \ell(H'_{n,1})$, etc. τ_n converges (up to isometries) to τ, the trapezoid in Figure 3 (left). We normalize so that we get actual convergence.

The Boundary Map: The arcs $H'_{n,1}, H'_{n,2}, D'_{n,1}, D'_{n,2}$ converge as sets to the line segments connecting their endpoints because $\ell(\partial\Omega_n) - \ell(\triangledown_n) \to 0$. Since I_n is length preserving, I_n converges uniformly to a linear isometry when restricted to each of $H_{n,1}, H_{n,2}, D_{n,1}, D_{n,2}$. Hence I_n converges on ∂M_{λ_n} to the piecewise linear isometry associated to the triangular Moebius band.

The Whole Map: We divide M_{λ_n} into 3 triangles as in Figure 4. (The points p and p' are discussed in §3.1.)

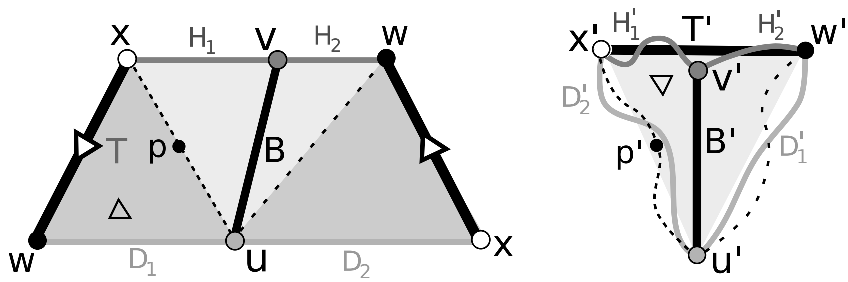

Figure 4: Figure 2 revisited

Consider the restriction of I_n to the left triangle $\triangle_n$. First, I_n is a linear isometry on $\overline{w_n x_n} = T_n$. Second, I_n converges to a linear isometry on the segment $\overline{w_n u_n} = D_{n,1}$. Third, I_n converges to a linear isometry on $\overline{x_n u_n}$ because $\|x'_n - u'_n\| \to \|x_n - u_n\|$ and I_n is distance non-increasing. In summary, I_n converges to a linear isometry on $\partial\triangle_n$. Since I_n is distance non-increasing, this implies that I_n converges to a linear isometry on $\triangle_n$. The same argument works for the other 2 triangles. Hence I_n converges to the piecewise linear isometry associated to the triangular Moebius band. This proves the Triangular Limit Theorem.

3 Discussion

3.1 Remarks on the Proofs

Bends and Mean Curvature: Concerning Proposition 2.1, here we sketch a proof that each $p \in U$ lies in a unique bend $\gamma \subset U$. This sketch follows my notes [**S4**]. Uniqueness is easy: If two bends intersect at p then U has zero mean curvature at p, a contradiction.

Existence: Since Ω has zero Gauss curvature and U has nonzero mean curvature, the differential $d\eta$ has 1-dimensional kernel and 1-dimensional image throughout U. Let $\gamma \subset U$ be the curve through p integral to kernel($d\eta$). The triple $\{\eta, \text{image}(d\eta), \text{kernel}(d\eta)\}$ defines an orthonormal frame along γ. First, η is constant along γ by integration. Second, γ is part of a foliation $\mathcal{F}$ by integral curves along which η is constant; this makes image($d\eta$) constant along γ. Hence kernel($d\eta$) is constant along γ, making γ a line segment.

Here is why γ cannot exit U: Suppose $q \in \overline{U} - U$ is some first exit point. Consider a unit vector field v along γ normal to γ but tangent to Ω. The line segments of $\mathcal{F}$ near γ cannot drastically separate from each other as they move "forwards" towards q because then they would "focus backwards" and intersect in U. Since η is constant along these segments, this geometry makes the ratio $\|d\eta_\zeta(v_\zeta)\|/\|d\eta_p(v_p)\|$ uniformly bounded away from 0 as $\zeta \to q$. This makes $\|d\eta_q(v_q)\| > 0$, contradicting the fact that $q \not\in U$.

An Explicit Homeomorphism: Concerning Lemma T, we construct an explicit homeomorphism between S^2 and $\overline{\Upsilon}$. Let X_θ denote the bend corresponding to $\theta \in \boldsymbol{R}/2\pi$. We identify ∂_+ and ∂_- respectively with the north and south pole of S_2. We parametrize $S^2 - \partial_\pm$ by (θ, ϕ), where $\theta \in \boldsymbol{R}/2\pi$ is the longitude and $\phi \in (0, \pi)$, the angle with the vector pointing to ∂_+, is the latitude. The antipodal map is $(\theta, \phi) \to (\pi + \theta, \pi - \phi)$ in these coordinates. We get a homeomorphism between $S^2 - \partial_\pm$ and Υ with the correspondence $(\theta, \phi) \leftrightarrow (X_{\theta-\phi}, X_{\theta+\phi})$. This conjugates the antipodal map to Σ and extends to a homeomorphism between S^2 and $\overline{\Upsilon}$. Thus, in the right coordinates, g and h are odd functions in the sense of the classic Borsuk-Ulam Theorem.

The Foliations: We revisit the Triangular Limit Theorem. Let p_n be the midpoint of $\overline{x_n u_n}$, shown in Figure 4. The point p'_n converges to the midpoint of $\overline{x'_n u'_n}$. The bend γ'_n through p'_n has its endpoints on $H'_{n,1}$ and $D'_{n,1}$, and this forces one endpoint of γ'_n to converge to x'_n and the other to u'_n. This means the pre-bend $\gamma_n = I_n^{-1}(\gamma'_n)$ converges to $\overline{x_n u_n}$. In other words, the left dotted line in Figure 4 is, for large n, quite close to a pre-bend. The same goes for the right dotted line. Hence the pre-bend foliation of M_{λ_n} converges to the pinstriping shown in Figures 1a (left) and 1b (left), and the bend foliation of Ω_n converges to the pinstriping shown in Figures 1a (right) and 1b (right).

3.2 Paths of Oriented Lines

Anton Izosimov and Sergei Tabachnikov independently suggested to me the following generalization of Lemma T.

Lemma 3.1 *Suppose* $\{L_t | \, t \in [0,1]\}$ *is a continuous family of oriented lines in* $\boldsymbol{R}^3$ *such that* $L_1 = L_0^{\text{opp}}$*, the same line as* L_0 *but with the opposite orientation. Then there exist parameters* $r, s \in [0,1]$ *such that* L_r *and* L_s *are perpendicular intersecting lines.*

Proof: This has the same proof as Lemma T, once we observe that our function h, defined in Equation 2, is more natural than we have let on. The points m_0, m_1 in the definition of h could be any points on u_0, u_1 and we would get the same result. g and h are invariants of pairs of oriented lines. ♠

Sergei also suggested to me a beautiful alternate formalism for Lemma T: the *dual numbers*. These have the form $x + \epsilon y$ where $x, y \in \boldsymbol{R}$ and $\epsilon^2 = 0$. Relatedly, the *dual vectors* have the form $\overrightarrow{a} + \epsilon \overrightarrow{b}$, where $\overrightarrow{a}, \overrightarrow{b} \in \boldsymbol{R}^3$ and again $\epsilon^2 = 0$. In this context, the dot product of two dual vectors makes sense as a dual number. See [**HH**] for an exposition.

Each oriented line $\ell \subset \boldsymbol{R}^3$ gives rise to a dual vector $\xi_\ell = \overrightarrow{a} + \epsilon \overrightarrow{b}$ where $\overrightarrow{a}$ is the unit vector pointing in the direction of ℓ and $\overrightarrow{b} = \ell' \times \overrightarrow{a}$. Here $\ell' \in \ell$ is any point. All choices of ℓ' give rise to the same $\overrightarrow{b}$; this vector is called the *moment vector* of ℓ. This formalism identifies the space of oriented lines in $\boldsymbol{R}^3$ with the so-called *Study sphere* consisting of dual vectors ξ such that $\xi \cdot \xi = 1$. The dual dot product $\xi_\ell \cdot \xi_m$ vanishes if and only if ℓ and m are perpendicular and intersect.

3.3 Related Topics

Square Peg: The Toeplitz Square Peg Conjecture asks if every continuous loop in the plane contains 4 points which make the vertices of a square. See [**Mat**] for a fairly recent survey. One can view a T-pattern as a collection of 4 points in the boundary of the Moebius band which satisfy certain additional constraints – e.g. they are coplanar. Put this way, a T-pattern is sort of like a square inscribed in a Jordan loop.

Quadrisecants: The idea for Lemma T is also similar in spirit for the idea developed in [**DDS**] concerning 4 collinear points on a knotted loop. These so-called *quadrisecants* play a role similar to Lemma T in getting a lower bound for the length of a knotted rope.

Folded Ribbon Knots: Elizabeth Denne pointed out to me the connection between paper Moebius bands and *folded ribbon knots*. Her paper with Troy Larsen [**DL**] gives a formal definition of a folded ribbon knot and has a wealth of interesting constructions, results, and conjectures. See also [**D**].

Folded ribbon knots are the objects you get when you take a flat cylinder or Moebius band, fold it into a knot, and then press it into the plane. Associated to a folded ribbon knot is a polygon, which comes from the centerline of the object. Even though the ribbon knot lies entirely in the plane, one assigns additional combinatorial data which keeps track of "infinitesimal" under and over crossings as in a knot diagram. So the associated centerline is really a knot (or possibly the unknot).

[**DL**, Corollary 25] proves our Main Theorem in the category folded ribbon Moebius bands whose associated centerline is a triangle. This is a finite dimensional problem. [**DL**, Conjecture 26], the analogue of the Halpern-Weaver Conjecture in the folded ribbon knot category, says that [**DL**, Corollary 25] is true without the very strong triangle restriction. The combination of our Main Theorem, the Triangular Limit Theorem, and smooth approximation as in [**HW**] implies [**DL**, Conjecture 26].

3.4 More Twists

One can make a *twisted cylinder* by taking a $1 \times \lambda$ strip of paper, giving it an even and nonzero number of twists, and then joining the ends together. There are two optimal limiting shapes which wrap a 1×2 strip 4 times around a right-angled isosceles triangle. In [**S3**] I prove that a twisted cylinder has aspect ratio greater than 2 and that any minimizing sequence converges on a subsequence to one of the two optimal models. This result also confirms the $n = 1$ case of [**DL**, Conjecture 39]. Noah Montgomery (private communication) independently came up with a different proof of the cylinder result.

Brienne Brown did some experiments with 3-twist paper Moebius bands and found two candidate optimal models which we call the *crisscross* and the *cup*. Both are made from a 1×3 strip of paper. The crisscross is planar and the cup is not. In [**BrS**] we conjecture that $\lambda > 3$ for an embedded multi-twisted paper Moebius band and that any sequence of minimizers converges on a subsequence to either the crisscross or the cup.

More recently, Aidan Hennessey [**H**] proved the fantastic result that one can make a cylinder or a Moebius band with any number of twists using a 1×6 strip. Extremely recently, following my lecture at UCLA on 8 Oct 2024, I told Jan Neinhaus about Hennessey's construction. The next day, Jan showed me how one can optimize and get $3\sqrt{3} + \epsilon$, for any $\epsilon > 0$, in place of 6. Jan's limiting shape is half a regular hexagon. We conjecture that $3\sqrt{3}$ is optimal in this context. Nobody has written about this yet.

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