The Optimal Paper Moebius Band (Informally)

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Abstract

This is an informal account of my solution of the Optimal Paper Moebius Band conjecture, written in a way that you should understand with just high school mathematics.

1 Introduction

This paper is an informal account of my paper [**S1**], which solves the Halpern-Weaver conjecture about optimal paper Moebius bands. The paper [**S1**] has a discussion of this problem and many references.

The purpose of this account is to give an informal account of my solution of the Optimal Moebius Band conjecture. The conjecture itself is so simple that practically anyone could understand it. The solution involves some advanced (but not too advanced) mathematics in a few spots, but it seems a shame not to try to make it so everyone can understand the solution as well. That is what I will try to do in these notes.

If you want to understand these notes, you should get out some paper, scissors, and tape. I'll frequently refer to a paper Moebius band, and if you are holding it in your hand you will have a better chance of knowing what I am talking about.

Here is how to make a paper Moebius band: Cut out a strip of paper, give it one twist, then tape the ends together. Figure 1 shows what you'll get.

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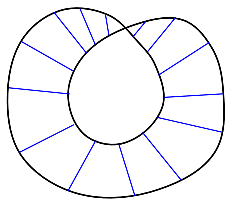

Figure 1: A paper Moebius band

For the sake of discussion suppose the width of your strip is always the same, namely one unit wide. If you start with a long strip, you can do the task easily. If you start with a shorter strip, you will find the task more difficult. How short can you go?

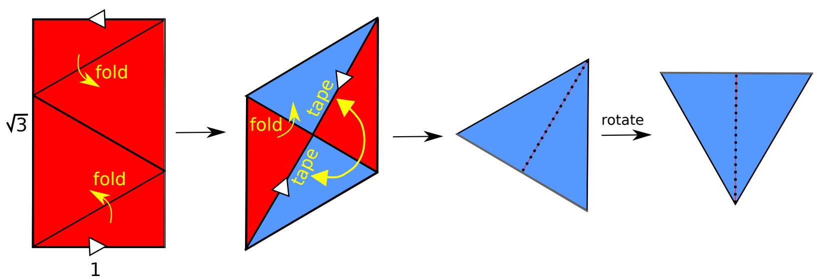

Figure 2: The optimal paper Moebius band?

Figure 2 shows a particularly beautiful example you can try to make. If you start with a $1 \times \sqrt{3}$ strip you can fold and (try to) tape it as indicated in Figure 2. The tape runs along the dotted line in the "inside" of the little triangular "wallet" you are making. I've never tried the taping, but in my mind it is taped.

Ideally speaking, the result is a perfectly flat Moebius band made out of paper. The final rotation in Figure 2 is not necessary but it highlights a kind of "T-pattern" you see in the picture, made from the top edge and dotted line. (This kind of T-pattern will be the star of our show.) Even if you stop reading this article now, you should fold up a strip as in Figure 1 and ponder it. It is a thing of great beauty and elegance.

The question is: Can you do better than this example? Informally speaking, can you turn a $1 \times \lambda$ strip into a paper Moebius band if $\lambda < \sqrt{3}$? That is what I proved: You cannot do better than $\sqrt{3}$. Also, if you make one out of a $1 \times \lambda$ strip and λ is only slightly larger than $\sqrt{3}$ then the result will look like Figure 1. In other words, the example in Figure 1 is uniquely the best one. All of this was conjectured by Halpern and Weaver in 1977, though Dmitry Fuchs tells me he thinks the conjecture is probably older.

2 Representing a Paper Moebius Band

Even if we use strips of the same length – I mean longer than $\sqrt{3}$, the length that forces Figure 2 – the paper Moebius band you make will not look precisely like one I would make. The problem is that it is too floppy. What you would like is a way to represent the thing by a planar diagram, much in the way that a planar map of the earth represents the earth itself. The left side of Figure 3 shows how this works.

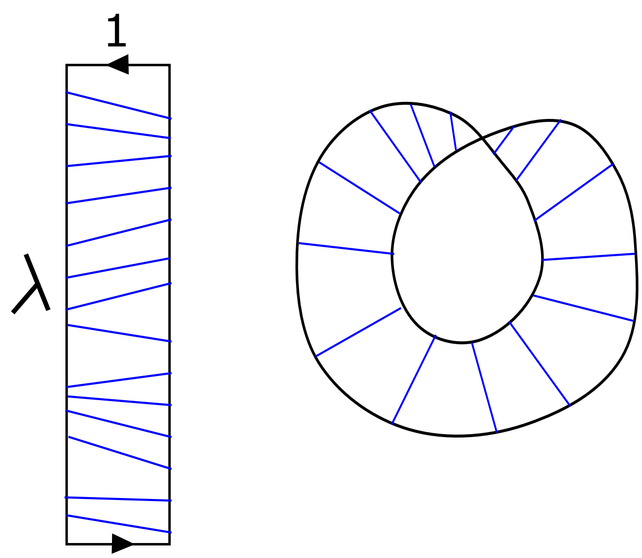

Figure 3: The rectangle (left) and the Moebius band (right).

The picture on the left is a rectangle, but the arrows on the top and bottom are supposed to explain how you tape up the ends to make the Moebius band on the right. The top left endpoint gets taped up to the botton right endpoint and the top right endpoint gets taped up to the bottom left endpoint.

You should imagine that every time your left hand touches a point on the left, your right hand touches the corresponding point on the right. Move your left hand around and your right hand follows along. If you were to have the right hand lead, then as your right hand approached the taped seam, your left hand would go to either the top or the bottom of the rectangle. Then as your right hand passed the seam you left hand would jump to the opposite side. In this way, we'll record spatial features of the actual Moebius band with a planar diagram.

3 The Bends

Pick up your paper Moebius band and hold it up close to your face. Now turn it around. If you turn it around just the right way, you'll see a kind of ridge that runs across it. This ridge is a perfectly straight line segment. Find a ridge and run your fingers along it. Feel its straightness. I'll call this ridge – meaning a perfectly straight line segment on the Moebius band that starts and ends on the boundary – a *bend*.

If you examine the Moebius band carefully, you will see that it has lots of bends. In fact, every point lies on a bend and the bends kind of fill out the Moebius band in the manner shown in Figures 1 and 3. The bends live on the actual 3D Moebius band. I'll call the corresponding line segments on the lefthand planar representation *pre-bends*. Each pre-bend has the same length as its corresponding bend.

Now imagine a kind of movie that works like this. Turn out the lights and imagine that one of the pre-bends is lit up like a neon light. Let us say it is yellow. The corresponding bend on the right is also lit up in neon blue. Now imagine that the pre-bend starts moving. You see this moving yellow neon light going up the rectangle and then, when it gets to the top, jumping back down in an instantaneous way.

What about on the right? You see this blue neon bend moving around in space, a flying spinning thing. There are no jumps to the image on the right, it just makes a continuous flight. When the blue bend makes one circuit it comes back to its original location. However, something interesting has happened, the two endpoints of the bend have switched. It has turned over, so to speak.

You can try this for yourself. Find a bend on your Moebius band and start mentally tracking it as it moves around. Use a pen or pencil to draw it. Now trace what happens when you go all the way around. You will see the turnover I am talking about.

Now think again about the movie of the blue neon light. At some point during the movie, the blue bend is turned "halfway". It is perpendicular to its original location. The bends themselves might not lie in the same plane, but if you looked down on them, so to speak, and then extended them into lines, the lines would be perpendicular. Let's call two bends *partners* if they are turned half way around from each other. Each bend has at least one partner, thanks to the turnover property. A bend could have more than one partner – this will come up again later on in the discussion.

4 T Patterns

Recall that partner bends are two bends whose directions are perpendicular. Let us call the pair of partner bends a *T-pattern* if they lie in the same plane. In Figure 2, the dotted line segment and the top edge make a T-pattern. The first step in my proof is showing that every sufficiently short paper Moebius band has a T-pattern. I'll sketch the proof in this section given some simplifying assumptions. The real proof in the paper takes care of these fine points.

Each pair of partner bends also defines a pair of parallel planes. One bend is contained in one plane and one bend is contained in the other plane. (One way to see these planes is to rotate space so that one of the partner bends points along the X-axis and the other points along the Y-axis. Then the two planes I am talking about are parallel to the XY plane.) Though these planes are parallel, they might not coincide. If they did coincide then the bends would make the T-pattern. Let's color these planes red and green. Let's say that the red plane is above the green plane.

For simplicity let's assume that each bend has a unique partner bend. Our blue bend, we have a second bend, which is the partner. Now we forget the blue bend and consider the red and green planes. As the blue bend moves, the red and green planes move. We have flying red and green planes, moving around like the wings of a dichrome butterfly. The wings stay parallel to each other and they sort of beat in and out. Now let's shorten the movie so that the initial blue bend does not go all the way around but rather only travels to the location of its partner and then stops. Let's rotate the picture so that the planes are horizontal at the start of the movie. What happens at the end of the movie? The two planes have interchanged! The plane that once contained the bend we are following now contains the partner.

How can this happen? There are two ways. One way is that the planes could stay distinct from each other and turn over. You should picture our butterfly doing a kind of corkscrew twist. The other way is that the planes must penetrate each other. At the instant of penetration they coincide, and this gives us a T-pattern.

We just have to rule out the turn-over option. Well, the planes start out horizontal and at the end are horizontal again. The turnover option takes a lot of length to execute – in fact more than $7\pi/12$ units of length, a number larger than $\sqrt{3}$. If we had a paper Moebius band whose length was anywhere near $\sqrt{3}$, the turnover option cannot happen and so the T-pattern exists.

5 The Triangle

Our T-pattern lives in some plane, and we'll rotate the picture so that we are looking down on it. Let B' and T' be the bends comprising the T-pattern. Consider the following two statements.

1. The line extending T' does not touch B'.

2. The line extending B' does not touch T'.

At least one of these statements is true, or they both could be true. In either case we can switch the names of B' and T', if needed, to arrange that the first statement is true. Now we rotate the picture so that B' is vertical and T' is horizontal. Since statement 1 is true, either B' lies entirely above the line extending T', or else entirely below it. I'll rotate the picture so that B' is below, which makes B' and T' look like a T.

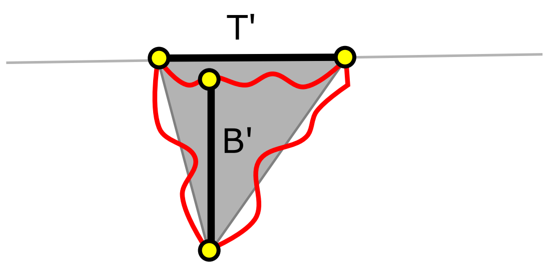

Figure 4: The T pattern and the triangle Δ.

Figure 4 shows a triangle Δ that is defined by the endpoints of T' and B'. How big is the perimeter of Δ. Well, B' and T' have length at least 1, and it turns out that the perimeter is as small as possible when both segments have length 1 exactly and they make a perfect T. That is, Δ should be an isosceles triangle of base 1 and height 1. The perimeter of this particular triangle is 2ϕ, twice the golden ratio.

Finally, Figure 4 shows how the boundary of our Moebius band sits with respect to the T-pattern. This red loop lives in space but it touches our plane at least at the vertices of the bends B' and T'. In particular, it contains the Δ vertices. Therefore, the boundary of the Moebius band is at least as long as the perimeter of Δ. At the same time, if you look at the rectangle picture in Figure 2 you can see that the length of the boundary is 2λ. We have therefore proved that $\lambda \geq \phi$. This is already an improvement over what had been known.

6 Cutting it Open

Let us cut open our Moebius band along B, the pre-bend corresponding to B'. So, pick up your paper Moebius band and cut it across, in a straight line, preferable at some angle to the boundary. What do you get? You get a trapezoid, as shown on the left side Figure 5. The colored vertices indicate the way you would tape this thing back together. The red vertices are taped together and so are the yellow ones. The blue and green vertices are just sitting there.

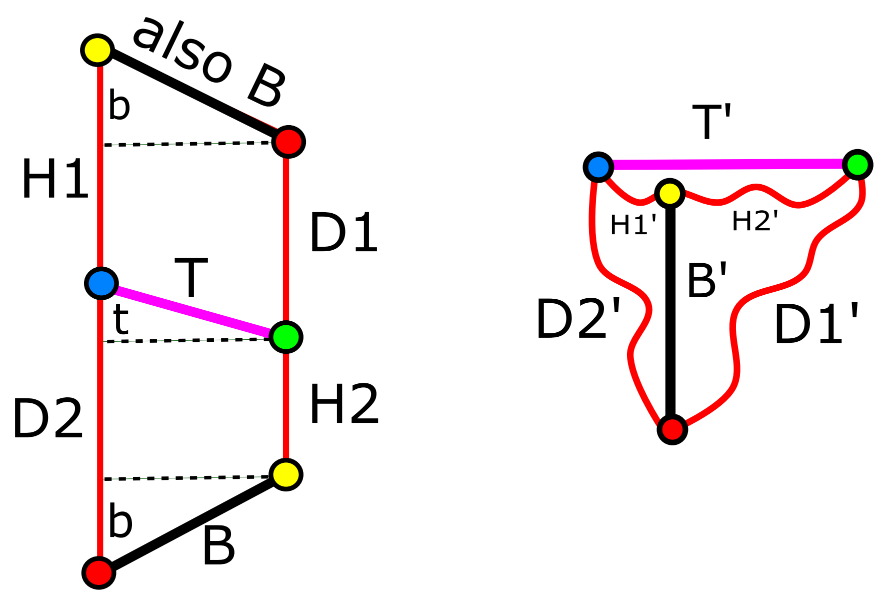

Figure 5: All the quantities you need to know.

I've drawn one of 4 basic pictures you could draw. In some of the other pictures, B slants downward on the bottom and upward at the top. In some of the other pictures, T slants upwards. It turns out that the calculations work out the same way for all possibilities. The formulas are all the same. This one is easiest on the eyes when you want to calculate things.

For example,

$$\ell(D_2) = \ell(H_2) + t + b. \tag{1}$$

If you define $(-t)$ to be the slope of T and b to be the slope of the bottom choice of B, then this formula works however the picture is drawn. Here ℓ stands for length. Another one of these formulas is

$$\ell(D_1) = \ell(H_1) + t - b. \tag{2}$$

7 Going in for the Kill

Here are some more formulas.

$$\ell(T') = \ell(T) = \sqrt{1+t^2}. \tag{3}$$

It turns out that the perimeter of a triangle (like Δ) having base $\sqrt{1+t^2}$ and height at least 1 is at least

$$P(t) = \sqrt{1+t^2} + \sqrt{5+t^2}. \tag{4}$$

Also the sum of the two diagonal sides of Δ is at least $\sqrt{5+t^2}$.

The length of the boundary of the Moebius band is

$$\ell(D_1) + \ell(D_2) + \ell(H_1) + \ell(H_2) = 2\ell(D_1) + 2\ell(D_2) - 2t. \tag{5}$$

As the same time the combined length of D_1 and D_2 is at least the length of the non-diagonal sides. Putting everything together. The length of the boundary is at least $2\sqrt{5+t^2} - 2t$. Remembering that this is 2λ, we have:

$$\lambda \geq \sqrt{5+t^2} - t. \tag{6}$$

To make the argument easier I'll just prove that $\lambda \geq \sqrt{3}$. I'll let you worry about why the case of equality cannot occur. I'll argue by contradiction. So, I'll assume that $\lambda < \sqrt{3}$ and something will go wrong. If $\lambda < \sqrt{3}$ then

$$\sqrt{5+t^2} - t < \sqrt{3}. \tag{7}$$

Adding t from both sides, squaring both sides, then simpifying, we discover that this inequality implies that $t > 1/\sqrt{3}$.

The function $P(t)$ gets larger as t increases. Since $t > 1/\sqrt{3}$, the perimeter of Δ is greater than $P(1/\sqrt{3})$. But – and here is the moment of truth – we have

$$P(1/\sqrt{3}) = 2\sqrt{3}. \tag{8}$$

Remembering that 2λ is at least the perimeter of Δ we see that $2\lambda > 2\sqrt{3}$. This is to say that $\lambda > \sqrt{3}$. This contradicts our assumption that $\lambda < \sqrt{3}$.

We're done here.

Acknowledgenents: I'd like to thank Selva Samuel for carefully reading this paper and pointing out typos.

[**S1**] R. E. Schwartz, *The Optimal Paper Moebius Band*, arXiv: 2308.12641