The Optimal Paper Moebius Band: A More Elementary Account

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Abstract

This paper gives a fairly elementary account of my proof of the optimal Moebius band conjecture.

1 Introduction

You make a paper Moebius band by taking a $1 \times \lambda$ strip of paper, giving it an odd number of twists, and then joining the length-1 ends together. The question is: How small can you make λ and still do this? This is an old problem, discussed in the introduction of the 1962 paper [**W**] of W. Wunderlich. (See [**T**] for an english translation of Wunderlich's paper.) The question seems implicit in the 1930 paper [**Sa**] of M. Sadowski. (See [**HF**] for an english translation of Sadowski's paper.) I guess that the question might go back even further than that. It might occur to anyone after they make a bunch of paper Moebius bands.

The question is treated in more detail in the 1977 paper of Benjamin Halpern and Charles Weaver. Halpern and Weaver conjecture in their paper that λ must be greater than $\sqrt{3}$. Sadowski's paper gives examples showing that you can take any $\lambda > \sqrt{3}$ and actually make a paper Moebius band, so the Halpern-Weaver Conjecture, if correct, really solves the whole problem.

I first learned about this *optimal paper Moebius conjecture* from the beautiful expository article written by Dmitry Fuchs and Sergei Tabachnikov,

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[**FT**, Chapter 14]. This is a chapter in their book, *The Mathematical Omnibus*. I recommend this work as an excellent introduction to the problem.

My paper [**S**] solves the optimal paper Moebius band conjecture. This is probably the first paper I have written that would be of interest to people even in high school. Scientific American [**C**], Popular Mechanics [**O**], Quanta Magazine [**Ha**], and a handful of other popular venues have covered the result. There are also some accounts on TikTok, Instagram, and Youtube. These popular accounts do not delve into the proof however.

The conjecture and its answer are pretty easy to state, so I thought it would be nice if someone without an extensive math background could follow the proof. The material here parallels the material in [**S**], but here I present things in a way that keeps the needed background to a minimum. I'll suggest a handful of experiments along the way. If you do the experiments you will get more out of this paper.

1.1 The Triangular Moebius Band

There is an experiment you can do that might lead you to the conclusion that $\sqrt{3}$ is the right bound.

Experiment 0: Make a paper Moebius band by starting with a long rectangular strip. Bring the ends together but don't tape them. Now slide the ends past each other, which in effect simulates making a shorter paper Moebius band. If you do this as much as possible, you will see a kind of triangular structure emerge.

Figure 1.1 shows the idealized limit of this procedure. The construction is based on a $1 \times \sqrt{3}$ rectangle.

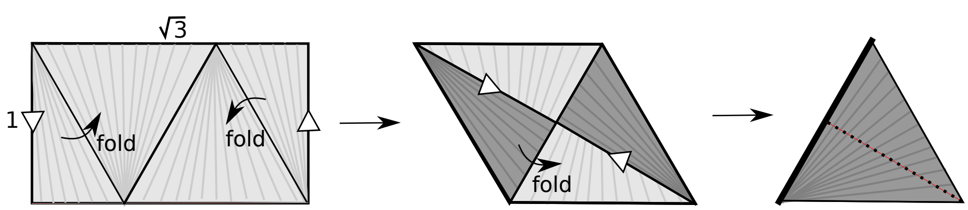

Figure 1.1: The triangular Moebius band

The strip in Figure 1.1 is lightly shaded on one side and darkly shaded on the other. You first fold in the flaps to make a rhombus, and then you fold the rhombus in half like a wallet. This cleverly joins the ends together. The dotted segment on the right runs through the middle of the triangular Moebius band and indicates where the ends are joined together. The bold segment on the right indicates the wallet fold. The dotted and bold segments together make what we will later call a *T pattern*.

Figure 1.2 shows a different view of the triangular Moebius band. This time we start with a symmetric trapezoid and fold the left and right sides over and under. This move brings the ends together at the top. This time the bold segment on the right indicates where the ends are brought together and the dotted segment again runs through the middle.

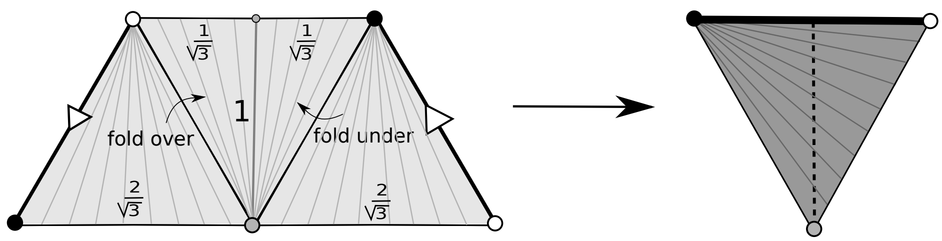

Figure 1.2: The triangular Moebius band: another view

The pinstriping in both figures indicates a collection of line segments, disjoint except at the endpoints, which remain straight during the folding process. This collection of line segments will play an important role in our definition of a paper Moebius band.

One thing strange about the triangular Moebius band is that you cannot make it exactly. If you made it exactly then the whole thing would fit exactly into an equilateral triangle. If you use a piece of paper that is very slightly longer, then you can make a Moebius band as intended. One part of our definition for what counts as a paper Moebius band is that it must be *embedded*. That is, the different parts of it cannot be completely pressed together.

Experiment 1: Make a triangular Moebius band for yourself. Draw the pinstriping on it. Find the T-pattern. Convince yourself that if you have a physical model, in which the sheets do not coincide, then actually you must have $\lambda > \sqrt{3}$.

1.2 The Moebius Accordion

Before giving our definition of a paper Moebius band, let's explore a wierd example that will not count. If you are just interested in folding a piece of paper up into a Moebius band, then you can actually do this for any $\lambda > 0$. Let me illustrate this with the case $\lambda = 1$. You start with a square piece of paper and fold it into a thin strip, like an accordion, using a zigzag pattern. You want to make an even number of folds, so that the strip is sectioned into an odd number of pieces.

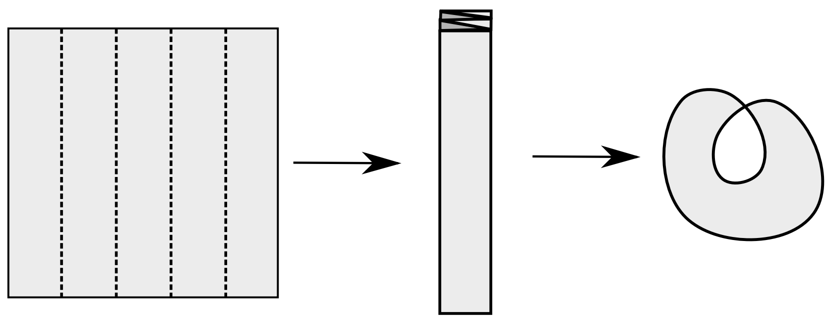

Figure 1.3: The Moebius accordion with 5 sections.

After making the thin strip, you then twist that thing into a Moebius band and join the ends together. The odd number of sections guarantees that the two pieces will fit together perfectly. This example really does make a Moebius band in space (rather than just a triangle) but it is still not embedded. The thin sections are all pressed exactly on top of each other.

Experiment 2: Make a Moebius accordion out of a 1×2 rectangle and then (using the other pair of opposite sides) out of a 2×1 rectangle. Convince yourself that you can do this for any rectangle you like.

The approximation you make will not quite lie flat. To really make this thing you will have to introduce small stretches or tears. The Moebius accordion not only fails to be an embedded paper Moebius band, but if you try to make approximations it is not possible to make it out of (theoretical) paper. Also, if you allow a monster like the Moebius accordion into the game then the question about the smallest λ is meaningless: Any λ will work.

1.3 Paper Moebius Bands

To formulate a meaningful question – and in particular to describe the Halpern-Weaver Conjecture – we need a definition of what a paper Moebius band actually is. The definition I give here is different than the traditional one given in [**S**], but what I call as a (smooth, embedded) paper Moebius band in [**S**] is covered under the definition given here. The translation between the definition given here and the traditional one goes back to the papers [**CL**] and [**HL**] in the 1950s.

Experiment 3: Make a paper Moebius band and examine it carefully. Notice that through each point you can draw a straight line segment that stays on the surface. Try to draw these line segments on the surface with a marker of some kind. Notice how the line segments cut across the Moebius band and have their endpoints in the boundary. Notice that the line sequence sweep through the paper Moebius band, making a kind of circle of line segments. The Moebius band bends around these line segments, and we will call them *the bends*

We denote the circle by S^1. A *circle of bends* is a loop of line segments in space. That is, for each angle $\theta \in S^1$, we have a line segment L_θ. Technically the segments should vary continuously with the parameter θ and also every two of them should be disjoint from each other. Again, we will call these line segments *bends*. Imagine you are in a dark room and you have a circular dial on the wall. The dial selects a bend which is glowing in the dark like a glow stick. As you turn the dial, the segment starts moving through space, as if an invisible person were swinging around the glow stick according to the commands of the dial.

Here is what it means for a surface to be a paper Moebius band.

- The surface is made out of a circle of bends. That is, the circle of bends sweeps through the surface. The endpoints of the bends lie in the surface boundary.
- When you cut the surface open along any of the bends, you can unwrap the resulting surface isometrically (i.e. without stretching or tearing) into the plane so that it becomes a symmetric trapezoid. The boundary opens up to be two parallel sides of the trapezoid and the two other sides come from the cut.

The intuitive idea is that when you make a paper Moebius band, these straight line segments automatically appear. The Moebius band is "bending along the bends".

Experiment 4: Make a paper Moebius band and cut it open along a bend. Notice how it falls open into a symmetric trapezoid. Experiment with cutting along different bends. The more perpendicular the bend is to the boundary, the closer your cut-open surface will resemble a rectangle.

Let's revisit the triangular Moebius band. The pinstriping in Figures 1.1 and 1.2 indicates the circle of bends. When you cut open along the dotted segment in Figures 1.1 and 1.2 you get a rectangle, which counts as a symmetric trapezoid. When you cut open along the bold segment in Figures 1.1 and 1.2 you get the symmetric trapezoid shown on the left side of Figure 1.2. I should say again that this definition does not quite work for the idealized triangular Moebius band because the bends are not all disjoint from each other. They trace over the same equilateral triangle three times. However, if you imagine that you have an approximation that is actually embedded in space, you will get an honest circle of bends.

After you cut the paper Moebius band open along the bend, there are two measurements you can make: The *height* is the distance between the parallel sides coming from the boundary. The *width* is the half the combined length of the two parallel sides coming from the boundary. The ratio width/height is the *aspect ratio*. In Figures 1.1 and 1.2 we have scaled so that the height is 1. We always scale this way. As we have already discussed, the aspect ratio in the idealized case is $\sqrt{3}$. For the approximations I have been talking about (where you can actually do the cutting) the aspect ratio is larger than $\sqrt{3}$.

Exierment 5: Make several paper Moebius bands using identical rectangular strips. Cut them open along different bends and compute the aspect ratio. Notice that you always get the same answer, namely the aspect ratio of the original rectangular strip. One thing to note is that if you cut the Moebius band open along any crosscutting straight line segment you will get a symmetric trapezoid. You don't even have to use a bend. So, in particular, you can do this experiment quite quickly.

1.4 The Main Result

The Main Theorem in [**S**] is that a paper Moebius band has aspect ratio greater than $\sqrt{3}$. I call this the Main Theorem in [**S**].

The other theorem in [**S**] is that if you have a paper Moebius band whose aspect ratio is close to $\sqrt{3}$ then the paper Moebius band itself is close to the triangular Moebius band. The closer you get to $\sqrt{3}$ the more the thing has to look like an equilateral triangle. In [**S**] I call this the Triangular Limit Theorem. The Triangular Limit Theorem provides the rigorous justification for the results of Experiment 0.

1.5 Further Reading

Since working out [**S**] there have been several other developments about paper Moebius bands. In [**BS**], my wife and I identified what probably are the shortest 3-twist optimal paper Moebius bands. These are ideal forms, like the triangular Moebius band, and they have aspect ratio 3. In [**H**], Aiden Hennessey proved the very surprising result that you can make a Moebius band with as many twists as you like with a 1×6.25 strip of paper.

You might also be interested in folded ribbon knots. These are like paper Moebius bands except that they are polygonal in nature and pressed into the plane. See [**D**] for a survey and [**DL**] for a wealth of conjectures about them. The triangular Moebius band is probably best interpreted as a folded ribbon knot. [**DL**, Corollary 25] is a special case of our Main Theorem in which the folded ribbon knot is assumed to have a triangular centerline. [**DL**, Conjecture 26] is a version of the Halpern-Weaver Conjecture in the language of folded ribbon knots. The Main Theorem of [**S**] incidentally solves this conjecture.

You might also be interested in rope knots. The general problem here is to tie a knot of a given kind with as short a piece of rope as possible. See [**CKS**] and [**DDL**] information about this.

My paper [**S**] has a more extensive bibliography discussing other work about paper Moebius bands and related topics.

1.6 Acknowledgements

In [**S**] I thank a ton of people for helping me think about the Optimal Moebius Band Conjecture. They are listed at the end of the introduction in [**S**].

Let me thank people here who specifically helped me with this more popular account. I thank Rachel Crowell and Kevin Hartnett who respectively wrote articles about [**S**] for Scientific American and Quanta Magazine. I thank Jim Hourihan, Killian Hourihan, Ander Petri-Hidalgo, Neil Schwartz (my brother), and Brienne Brown (my wife) for discussions about this informal version. I would also like to thank Ana and Mia Jain, two 6-year-old twins who did an extensive science project based on [**S**]. Their project gave me the good idea of enhancing the exposition here with a list of suggested experiments.

2 Four Optimization Problems

In this chapter I will discuss 4 "optimization problems". We put certain constraints on a shape we are interested in, and then we try to find the best shape according to some measurement. The results are stated in terms of inequalities of the form "The measurement for any shape satisfying the constraints is at least ..." In the next chapter I will relate the last of the problems, the Coupled Circuit Problem, to the Optimal Moebius Problem. Here is one piece of notation. In general, $|\gamma|$ denotes the length of a curve γ.

2.1 The Triangle Problem

Let $\bigtriangledown$ be a triangle with a horizontal base of length x, and a height of y. Let $\vee$ denote the two non-horizontal sides. The measurement $|\vee|$ is the sum of the lengths of the two non-horizontal sides. See Figure 2.1. We claim that

$$|\vee| \geq \sqrt{x^2+4y^2}. \tag{1}$$

The best shapes in this case are the isosceles triangles.

Here is the argument. Let x_1 and x_2 be the two vertices of the horizontal side. Let q be the third vertex. Let x_2' be the point you get by reflecting x_2 in the horizontal line through q.

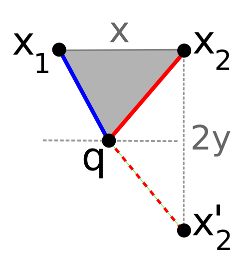

Figure 2.1: The Triangle Problem.

We have

$$|\vee| = |\overline{x_1q}| + |\overline{qx_2}| = |\overline{x_1q}| + |\overline{qx_2'}| \geq |\overline{x_1x_2'}| = \sqrt{x^2+4y^2}.$$

The first equality is just the definition of $|\vee|$. The second equality comes from symmetry: The segments $\overline{qx_2}$ and $\overline{qx_2'}$ have the same length. The inequality comes from the fact that the two segments $\overline{x_1q}$ and $\overline{qx_2'}$ make a path joining x_1 to x_2', and the shortest possible path connecting these points is just the segment $\overline{x_1x_2'}$. The last equality is the Pythagorean Theorem.

2.2 The Planar Circuit Problem

Our next problem builds on the Triangle Problem but recasts it in a slightly different way. You should compare Figures 2.1 and 2.2 and see that really they are not that different from each other.

Suppose $X = \overline{X_1X_2}$ and $Y = \overline{Y_1Y_2}$ are respectively horizontal and vertical line segments having lengths

$$|X| = x = \sqrt{1+t^2}, \qquad |Y| = y \geq 1. \tag{2}$$

For now, writing $x = \sqrt{1+t^2}$ is just a complicated way of saying that $x \geq 1$, but in the next section the quantity t will have an additional meaning. We are just warming up at this point.

Suppose also that Y is a vertical line segment that lies completely underneath the horizontal line extending X. Finally, suppose γ is a continuous loop that successively connects the points X_1, Y_1, X_2, Y_2. Figure 2.2 shows all this.

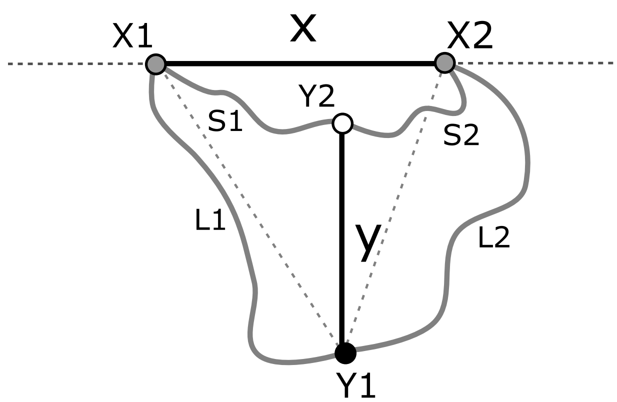

Figure 2.2: A circuit γ connecting the endpoints of X and Y.

The 4 arcs L_1, L_2, S_2, S_1 comprise γ. Let $|L| = |L_1| + |L_2|$ and likewise $|S| = |S_1| + |S_2|$. Each arc is at least as long as the straight line segment with the same endpoints. Hence $|S| \geq |X|$ and $|L| \geq |\vee|$, where $\vee$ is the two non-horizontal sides of the triangle $\bigtriangledown$ having vertices X_1, X_2, Y_1. This triangle has base x and height greater than 1. Equations 1 and 2 give:

$$|L| > \sqrt{5+t^2}, \qquad |\gamma| = |S| + |L| > \alpha(t) := \sqrt{1+t^2} + \sqrt{5+t^2} \tag{3}$$

Our notation means that we are defining $\alpha(t) = \sqrt{1+t^2} + \sqrt{5+t^2}$ and we have $|\gamma| > \alpha(t)$.

2.3 The Coupled Planar Circuit Problem

This problem builds on the planar circuit problem but adds a new constraint to the problem. We keep the same notation from the planar circuit problem.

This new constraint "couples" our loop γ to the segment X. What this means is that we add some new constraint that forces γ and X to depend on each other. This new constraint involves the variable t from above. Here it is:

$$|S| = |L| - 2t. \tag{4}$$

Equations 3 (left) and 4 give another bound:

$$|\gamma| = |S| + |L| = 2|L| - 2t > \beta(t) := 2\sqrt{5 + t^2} - 2t. \tag{5}$$

Combining Equations 3 and 5, we have $|\gamma| > \mu(t) := \max(\alpha(t), \beta(t))$.

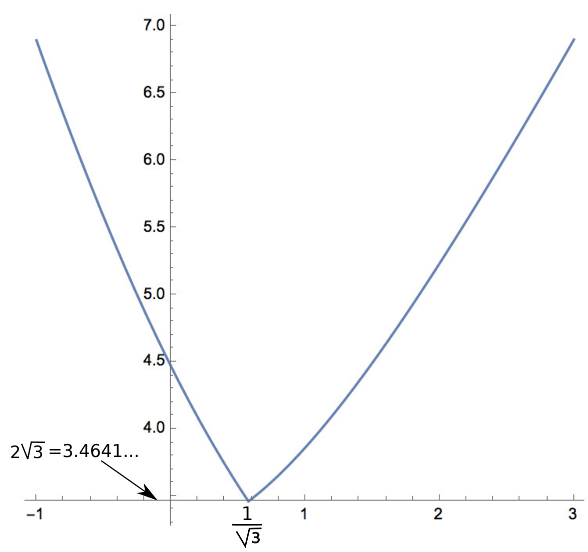

Figure 2.3: A plot of $\mu(t)$ for $t \in [-1, 3]$.

Figure 2.3 shows a plot of $\mu(t)$. The minimum occurs when $t = 1/\sqrt{3}$, and the minumum value is $2\sqrt{3}$. Therefore, $|\gamma| > 2\sqrt{3}$. This number should be familiar from our main result.

There is more we can say here. if $|\gamma| \approx 2\sqrt{3}$ then $t \approx 1/\sqrt{3}$ and $x \approx 2/\sqrt{3}$, and $\bigtriangledown$ has height ≈ 1 and L_1, L_2, S_1, S_2 are all nearly line segments. Hence γ closely follows an equilateral triangle. This should resonate with our second result, the Triangular Limit Theorem.

2.4 The Coupled Circuit Problem

This is almost the same problem as the previous one. This time we think of our plane as the XY-plane sitting in space. We keep the segments X and Y as before but this time we allow our loop γ to move in space, above and below the XY-plane. All we require is that the endpoints of the 4 arcs L_1, L_2, S_2, S_1 comprising γ are again X_1, Y_1, X_2, Y_2. Figure 2 again depicts the situation, except that you should imagine you are looking down on the XY plane from space. This problem has the same analysis as the planar version. Once again we get $|\gamma| > 2\sqrt{3}$. Also, if $|\gamma| \approx 2\sqrt{3}$ then γ nearly traces out an equilateral triangle.

Let us relate this to paper Moebius bands. Let M be a paper Moebius band of aspect ratio λ. As always, we scale so that M has height 1. Let γ be the loop which is the boundary of M. Recall that 2λ is exactly the length of the boundary of M. That is, $2\lambda = |\gamma|$. We aim to recognize γ as a loop that arises in the coupled circuit problem. Once we recognize this, the solution of the coupled circuit problem tells us that $|\gamma| > 2\sqrt{3}$. This tells us that $\lambda > \sqrt{3}$ and proves the Main Theorem.

We can say more. If $\lambda \approx \sqrt{3}$, then $|\gamma| \approx 2\sqrt{3}$ and so, by our analysis above, γ must nearly trace out an equilateral triangle. This gives us the Triangular Limit Theorem.

3 Recognizing a Coupled Circuit

3.1 T Patterns

Recall that a paper Moebius band M has a circle of bends sweeping through it. A T-*pattern* on M is a pair of these bends which lie in the same plane and point in perpendicular directions. The dotted and bold segments in Figures 1.1 and 1.2 make a T-pattern.

Experiment 6: Make a paper Moebius band and find a T-pattern on it.

I'll prove below that every paper Moebius band has a T-pattern. Let us assume for now that M has a T-pattern. We can rotate M in space so that the bends X, Y of the T-pattern, and the loop γ, are situated just as in the Coupled Circuit Problem. X and Y cut across M and so have length at least 1. This gives Equation 2.

To derive Equation 4 we cut open M along Y and flatten it out in the plane. We get a symmetric trapezoid τ. Figure 4 shows one of several possible ways τ could look, depending on how X and Y slant. The labels match Figure 2.2. (The repeat of Figure 2.2 included for convenience, is not quite drawn to scale.)

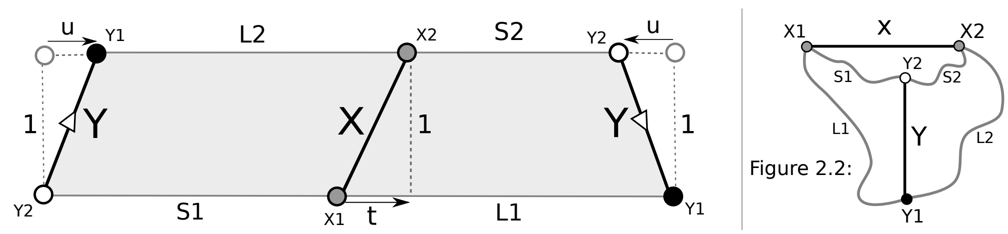

Figure 3.1 The symmetric trapezoid τ.

The left and right sides of τ get the same labels because on M they are joined together. Since $|X| = \sqrt{1+t^2}$ the Pythagorean Theorem tells us that t equals the horizontal displacement of the endpoints of X. Therefore

$$|S_1| + t = |L_2| + u, \qquad |S_2| + u = |L_1| - t. \tag{6}$$

We get Equation 4 by adding these equations together and simplifying. For the other possible pictures of τ, in which X and/or Y slant the other way, the signs of t and/or u change but we get Equation 4 in all cases.

Having recognized γ as a loop that arises in the Coupled Circuit problem, we get the two theorems in [**S**], as we discussed at the end of the last chapter.

3.2 The Sphere

The rest of this paper just amounts to proving that a paper Moebius band always has a T-pattern. We first discuss 3 views of the sphere.

First View: We let S^2 denote the sphere of unit vectors in space. In other words, S^2 is the set of points (x, y, z) with $x^2 + y^2 + z^2 = 1$.

Second View: The north pole and south pole respectively are the points $P_+ = (0, 0, +1)$ and $P_- = (0, 0, -1)$. We can get a nice coordinate system for S^2 which works away from P_+ and P_-. We use a pair of angles (θ, ϕ) which denote longitude and latitude. Here θ, the longitude, takes values in S^1, the circle. The latitude ϕ takes values in the interval $(0, \pi)$. That is, $0 < \phi < \pi$. This angle represents the angle that the given point makes with P_+. When $\phi \approx 0$ we are at at a point near P_+. When $p \approx \pi$ we are at a point near P_-. When $\phi = 0$ we would be at P_+, but then θ is not well defined. The same problem happens when $\phi = \pi$. This is why we say that the coordinate system works away from P_+ and P_-.

Third View: We can think of S^2 as the set of arcs of the circle S^1. For each arc of S^1 we can find a pair (θ, ϕ) so that the arc goes counterclockwise from the angle $\theta - \phi$ to the angle $\theta + \phi$. Angles in S^1 are only defined up to multiples of 2π. So, for instance, if $\theta = 3\pi/2$ and $\phi = 3\pi/2$, then $\theta + \phi = 3\pi$, whcih is the same as π. When $\phi \approx 0$ the arc is quite short. When $\phi \approx \pi$ the arc wraps almost all the way around S^1. Given this geometric behavior, it makes sense to interpret P_+ as the *empty set* and P_- as all of S^1. So, if we include the empty set and the whole circle as arcs, then S^2 is exactly the set of arcs of S^1.

The Antipodal Map: There is a symmetry of S^2 called the *antipodal map*. In our first coordinate system, the antipodal map exchanges the diametrically opposed points (x, y, z) and $(-x, -y, -z)$. In our second coordinate system, the map exchanges the points (θ, ϕ) and $(\pi + \theta, \pi - \phi)$. Note that these two points correspond to complementary arcs: (θ, ϕ) describes the arc that starts at $\theta - \phi$ and ends at $\theta + \phi$ whereas $(\pi + \theta, \pi - \phi)$ describes the arc that starts at $\theta + \phi$ and ends at $\theta - \phi$. The two arcs connect the same two points but trace out opposite sides of these two points. This description even works for the "arcs" corresponding to P_+ and P_- because the complement of the empty set is S^1 and *vice versa*. The antipodal map on S^2 corresponds to taking complementary arcs in S^1.

Let us call the antipodal map A. Here is a formula for A in each of our coordinate systems.

$$\begin{aligned} A(x,y,z) &= (-x,-y,-z), \\ A(\theta,\phi) &= (\pi+\theta,\pi-\phi), \\ A(\theta_0,\theta_1) &= (\theta_1,\theta_0). \end{aligned} \tag{7}$$

In the third equation, we are describing an arc of S^1 by its starting and endpoint point. We think of starting at θ_0 and moving counter-clockwise around until reaching θ_1. The complementary arc has the same endpoints but they come in the opposite order.

3.3 Continuous Functions

A *function* on S^2 is an assignment of each point of p a value $f(p)$. For instance, if S^2 is the surface of the earth, then $f(p)$ might be the current temperature at p. The function f is called *continuous* if small changes in p result in small changes in f. So, if we want to guarantee that f changes by less than any given amount, say 10^{-100}, then as long as we pick a point p' sufficiently close to p the difference between $f(p)$ and $f(p')$ will be less than 10^{-100}.

Here is a more formal definition of a continuous function: For all p on S^2 and for all $\epsilon > 0$ there is a $\delta > 0$ so that if we move p by more less than δ the value $f(p)$ changes by less than ϵ. In the less formal description given above, ϵ was 10^{-100}.

To recognize a T-pattern in the Moebius band M, we are going to define two functions g and h of a pair (u_0,u_1) of bends. We will interpret g and h as functions on S^2 and they will be continuous. When $g(u_0,u_1) = 0$ the pair (u_0,u_1) will point in perpendicular directions. When $h(u_0,u_1) = 0$ the pair (u_0,u_1) will lie in the same plane. So, if we have a common zero $g(u_0,u_1) = h(u_0,u_1) = 0$ then we have our T-pattern.

Let me explain the power of continuity. Let's suppose that $g(p) < 0$ and $g(q) > 0$. Then we can consider any path from p to q on the sphere. Since g switches from positive to negative along the path, there must be some place where $g = 0$ along the path. We don't know where this zero is, but the continuity guarantees it. If we want a *common zero* to the two functions, we need a more sophisticated argument.

3.4 The Borsuk-Ulam Theorem

A function f on S^2 is called *odd* if $f(A(p)) = -f(p)$. That is, f takes on opposite values at antipodal points. The Borsuk-Ulam Theorem says that two odd functions g and h have a common zero. This is exactly what we are looking for. (We have not defined g and h yet, nor shown that they are odd, but that is coming.) Now I will sketch a proof of the Borsuk-Ulam Theorem.

For simplicity assume the following conditions:

$$g(P_+) = +1, \qquad g(P_-) = -1, \qquad h(P_+) = h(P_-) = 0. \tag{8}$$

This isn't really much of a restriction and anyway our functions g and h, defined below, will have these conditions.

We bundle together g and h into a single *map* $F = (g, h)$. So, for each point p on the sphere $F(p) = (g(p), h(p))$ is a point in the plane. We want to see that there is some point p such that $F(p) = (0, 0)$. We will argue by contradiction. We will suppose that there is no such point and then we will end up at an impossible conclusion. The only way out of the contradiction is that there must be a common zero.

Consider a line ℓ_θ of longitude on S^1. This path starts at P_+, goes through the point $(\theta, \pi/2)$ on the equator, and continues on to P_-. Consider the image $F(\ell_\theta)$ in the plane. This path starts out at $(1, 0)$ and ends at $(-1, 0)$ and (by assumption) misses $(0, 0)$. If you stand at $(0, 0)$ and watch $F(\ell_\theta)$ as it goes, your next will twist a half-integer number of times, either clockwise or counter-clockwise. Let's say that this number $N(\theta)$ is positive if the overall net twisting is more counterclockwise and negative if the overall net twisting is more clockwise. If we move θ just a little bit, then $N(\theta)$ does not change. This function is both continuous and half-integer valued. But then $N(\theta)$ is always the same value, no matter the value of θ. In particular, $N(0) = N(\pi)$.

Consider ℓ_π. We have $\ell_\pi = A(\ell_0)$ except that we have to verse the direction of $A(\ell_0)$ so that it runs from P_+ to P_-. Since g and h are odd functions, $F(A(\ell_0)) = -F(\ell_0)$. Geometrically, $F(A(\ell_0)$ is obtained by rotating $F(\ell_0)$ 180 degrees around the origin. As $F(A(\ell_0))$ goes from $(-1, 0)$ to $(1, 0)$ it winds the same number of times as does $F(\ell_0)$. But remember that $F(\ell_\pi)$ is obtained by reversing the direction of $F(A(\ell_0))$. This means that $N(\pi) = -N(0)$. This is a contradiction. The only way out that is that we cannot always define $N(\theta)$. This means that there is some p so that $F(p) = (0, 0)$. This completes the proof of the Borsuk-Ulam Theorem.

3.5 Crash Course on Vectors

To find our T-pattern, it remains to define the functions g and h, show that they are odd, use the Borsuk-Ulam Theorem, and then interpret the result geometrically. We need to collect a few more preliminary ideas before doing all this.

To match tradition, we say that a *vector* is a point in space. A *unit vector* is a point in S^2. What is the difference between points and vectors? Well, nothing really, except that one often thinks of a vector as being more like a direction in space. We usually denote vectors like this: $\overrightarrow{v}$. There are two operations on vectors which we now describe.

The Dot Product: Given vectors $\overrightarrow{v}_0 = (x_0, y_0, z_0)$ and $\overrightarrow{v}_1 = (x_1, y_1, z_1)$ and the dot product is a number:

$$\overrightarrow{v}_0 \cdot \overrightarrow{v}_1 = x_0x_1 + y_0y_1 + z_0z_1. \tag{9}$$

When the vectors are nonzero, their dot product equals 0 exactly when they are perpendicular. When the vectors are unit vectors, the dot product is ± 1 exactly when $\overrightarrow{v}_1 = \pm\overrightarrow{v}_0$. The dot product is symmetric and bi-linear:

$$\overrightarrow{v}_0 \cdot \overrightarrow{v}_1 = \overrightarrow{v}_1 \cdot \overrightarrow{v}_0, \qquad (a\overrightarrow{v}_0 + \overrightarrow{w}_0) \cdot \overrightarrow{v}_1 = a(\overrightarrow{v}_0 \cdot \overrightarrow{v}_1) + \overrightarrow{w}_0 \cdot \overrightarrow{v}_1.$$

You can see these properties directly from the formula.

The Cross Product: Given vectors $\overrightarrow{v}_0 = (x_0, y_0, z_0)$ and $\overrightarrow{v}_1 = (x_1, y_1, z_1)$ and the cross product is another vector:

$$\overrightarrow{v}_0 \times \overrightarrow{v}_1 = (y_0z_1 - y_1z_0, z_0x_1 - z_1x_0, x_0y_1 - x_1y_0.) \tag{10}$$

For nonzero vectors, the cross product vanishes exactly when the vectors are multiples of each other. When the cross product is nonzero, it is perpendicular to both vectors. The cross product is anti-symmetric and bi-linear:

$$\overrightarrow{v}_0 \times \overrightarrow{v}_1 = -\overrightarrow{v}_1 \times \overrightarrow{v}_0, \qquad (a\overrightarrow{v}_0 + \overrightarrow{w}_0) \times \overrightarrow{v}_1 = a(\overrightarrow{v}_0 \times \overrightarrow{v}_1) + \overrightarrow{w}_0 \times \overrightarrow{v}_1.$$

You can also see these properties directly from the formulas.

We will use the dot product and the cross product to define our magic functions g and h.

3.6 The Entailing Operation

We need one more ingredient before we introduce our magic functions g and h. We need to relate the circle of bends to vectors.

Each bend u in our circle of bends has a pair of unit vectors which are parallel to it. We denote these by $\overrightarrow{u}$ and $-\overrightarrow{u}$. There is no way to prefer one over the other. In this way, we assign a pair of unit vectors to each point in the circle S^1. We call both these vectors *orientations* of the bend. A point θ in S^1 corresponds to the bend L_θ, and then the two orientations are $\pm\overrightarrow{L}_\theta$.

Suppose we have an arc of S^1, say (θ_0, θ_1). If we choose an orientation $\overrightarrow{L}_{\theta_0}$ then there is exactly one way to choose an orientation for L_{θ_1} so that the orientations vary continuously along the arc. As we move along the arc we just avoid making any sudden jumps. We only have two orientations at each point, so there is a clear choice that makes the orientations vary continuously. We write $\overrightarrow{L}_{\theta_0} \rightsquigarrow \overrightarrow{L}_{\theta_1}$ when these orientations are related as we have just been describing. The symbol $\rightsquigarrow$ means *entails*. Let us simplify our notaton by setting $u_0 = L_{\theta_0}$ and $u_1 = L_{\theta_1}$.

Suppose that $\overrightarrow{u}_0 \rightsquigarrow \overrightarrow{u}_1$. Here are the key properties of our entailing operation.

1. $-\overrightarrow{u}_0 \rightsquigarrow -\overrightarrow{u}_1$.

2. $-\overrightarrow{u}_1 \rightsquigarrow -\overrightarrow{u}_0$. This is because we are on a Moebius band!

3. If our arc is short, corresponding to a point near P_+, then

$$\overrightarrow{u}_1 \approx \overrightarrow{u}_0.$$

   This is because the orientation hardly moves at all.

4. If our arc is almost all of S^1, corresponding to a point near P_-, then

$$\overrightarrow{u}_1 \approx -\overrightarrow{u}_0.$$

   This is because we are on a Moebius band!

Experiment 7: Make a paper Moebius band and draw some orientations on the bend. Try to check that the 4 above properties of the entailing operation are true for some examples of pairs of bends.

3.7 The Magic Functions

Now we can define or magic functions g and h and find the T-pattern.

Let (u_0, u_1) be a pair of bends. Again, we think of (u_0, u_1) as describing an arc of S^1, which in turn gives us a point of S^2. Let m_0 and m_1 respectively be the midpoints of u_0 and u_1. We define

$$g(u_0, u_1) = \overrightarrow{u}_0 \cdot \overrightarrow{u}_1, \qquad h(u_0, u_1) = (m_0 - m_1) \cdot (\overrightarrow{u}_0 \times \overrightarrow{u}_1). \tag{11}$$

Here $\overrightarrow{u}_0 \rightsquigarrow \overrightarrow{u}_1$. Since $-\overrightarrow{u}_0 \rightsquigarrow -\overrightarrow{u}_1$, we would get the same values starting with $-\overrightarrow{u}_0$. In other words, our functions are well defined. The functions are continuous because the bends vary continuously in our circle of bends.

We have not yet defined g and h on P_+ and P_- but Properties 3 and 4 of the entailing operation tell us that we should define $g(P_+) = 1$ and $g(P_-) = 1$ and $h(P_+) = h(P_-) = 0$. Once we do this, g and h are continuous functions on S^2.

Let us now check that g and h are odd functions on S^2.

$$g(u_1, u_0) = \overrightarrow{u}_1 \cdot (-\overrightarrow{u}_0) = -\overrightarrow{u}_0 \cdot \overrightarrow{u}_1 = -g(u_0, u_1),$$

$$h(u_1, u_0) = (m_1 - m_0) \cdot (\overrightarrow{u}_1 \times (-\overrightarrow{u}_0)) = (m_1 - m_0) \cdot (\overrightarrow{u}_0 \times \overrightarrow{u}_1) = -h(u_0, u_1).$$

By the Borsuk-Ulam Theorem there is some p in S^2 such that

$$g(p) = h(p) = 0.$$

Note that $p \neq P_\pm$ because $g(P_\pm) = \pm 1$. This means that p corresponds to an honest pair of unequal bends (X, Y). That is

$$g(X, Y) = h(X, Y) = 0.$$

Since $g(X, Y) = 0$ we have $\overrightarrow{X} \perp \overrightarrow{Y}$. In other words, the bends point in perpendicular directions. Let $\overrightarrow{n} = \overrightarrow{X} \times \overrightarrow{Y}$. Since $h(X, Y) = 0$, the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ and $m_X - m_Y$ are all perpendicular to $\overrightarrow{n}$. (Here m_X is the midpoint of X and m_Y is the midpoint of Y.) But then, using the bi-linearity of the cross product, every vector of the form $v_X - v_Y$, where v_X is a point of X and v_Y is a point of Y, is perpendicular to $\overrightarrow{n}$. This implies that X and Y lie in the same plane perpendicular to $\overrightarrow{n}$. In short, X and Y are co-planar. This shows that (X, Y) makes a T-pattern.

Our proof is done.

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