Research Announcement: Unbounded Orbits for Outer Billiards

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Abstract

In [**S**] we proved that the outer billiards system defined on the Penrose kite has an unbounded orbit. In this article we will sketch some of the main ideas in the proof, and describe in detail a very convincing computer demonstration of our result.

1 Introduction

Outer billiards is a basic dymamical system which serves as a toy model for celestial mechanics. We refer the reader to Sergei Tabachnikov's book [**T**], and also the survey [**DT**], for an exposition of the subject. To define an outer billiards system, one starts with a bounded convex set $S \subset \boldsymbol{R}^2$ and considers a point $x_0 \in \boldsymbol{R}^2 - S$. One defines x_1 to be the point such that the segment $\overline{x_0x_1}$ is tangent to S at its midpoint and S lies to the right of the ray $\overrightarrow{x_0x_1}$. This construction is defined for almost every point x_0. Whenever defined, the iteration $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow ...$ is called the *outer billiards orbit* of x_0.

B.H. Neumann [1] introduced outer billiards during some lectures for popular audiences given in the 1950s. See, e.g. [**N**]. J. Moser popularized the construction in the 1970s. Moser [**M**, p. 11] asks whether there is a shape on which the outer billiards system has an unbounded orbit, though he attributes the question to B.H. Neumann *circa* 1960.

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[1] My information on this comes from 1999 email correspondence between Bernhard Neumann and Keith Burns, and also from Bernhard's son Walter.

There have been several results related to this question. Moser [**M**] sketches a proof, inspired by K.A.M. theory, that outer billiards on S has all bounded orbits provided that ∂S is at least C^6 smooth and positively curved. R. Douady gives a complete proof in his thesis, [**D**]. See [**B**] for related work. In [**VS**], [**Ko**], and (later, but with different methods) [**GS**], it is proved that outer billiards on a *quasirational polygon* has all orbits bounded. This class of polygons includes polygons with rational vertices and also regular polygons. Genin [**G**] shows that all orbits are bounded for the outer billiards systems associated to trapezoids. He also makes a brief numerical study of a particular irrational kite based on the square root of 2, observes possibly unbounded orbits, and indeed conjectures that this is the case.

In [**S**] we prove

Theorem 1.1 *Outer billiards on the Penrose kite has an unbounded orbit.*

The Penrose kite is the convex quadrilateral that comes up in the famous Penrose tiling. This quadrilateral is affinely equivalent to the quadrilateral with vertices

$$(0,1); \quad (-1,0); \quad (0,-1); \quad (A,0); \qquad A = \mathrm{dec}(\sqrt{5}) \sim 0.2360679977. \quad (1)$$

Here $\mathrm{dec}(x) \in (0,1)$ denotes the decimal part of $x \not\in \boldsymbol{Z}$.

We actually prove quite a bit more about outer billiards on the Penrose kite. We show that there is an uncountable number of unbounded orbits, each of which returns densely to a certain Cantor set according to the dynamics of the 2-adic adding machine. This fine structure is a consequence of the quasi-self-similarity of something we call the *arithmetic graph*. The arithmetic graph behaves like the tilings studied in [**Ke**].

In this paper we will explain the arithmetic graph and give precise instructions for how to draw it. Our construction works for any irrational $A \in (0,1)$ in Equation 1 and so we will explain it in this generality. The interested reader can make experiments using other values of A (most notably $A = \mathrm{dec}(\sqrt{2})$) and see that the unboundedness phenomenon is quite robust. At the moment we only have a proof for $a = \mathrm{dec}(\sqrt{5})$, but we are thinking about a more general proof.

We wrote an extensive graphical user interface, *Billiard King*, to explore this problem, and all our ideas were discovered while playing with this program. We encourage the reader to download the Java based program (www.math.brown.edu/∼res/BilliardKing/BilliardKing.tar) and play with it.

2 The Arithmetic Graph

Let $A \in (0,1)$ be irrational. All our constructions are based on A. Define $T : \boldsymbol{Z}^2 \to \boldsymbol{R}$ via the formula

$$T(x,y) = 2Ax + 2y + \frac{1-A}{2}. \tag{2}$$

When $A = \text{dec}(\sqrt{5})$ our unbounded orbit is the point

$$(T(0,0),1). \tag{3}$$

Lemma 2.1 *The outer billiards map is entirely defined on any point of the form*

$$T(\boldsymbol{Z}^2) \times \{\pm 1\}$$

In particular, the entire orbit of the point in Equation 3 is defined.

Proof: Let L denote the family of horizontal lines in $\boldsymbol{R}^2$ whose y-coordinates are odd integers. The outer billiards map preserves L. The only points where the the first iterate of the outer billiards map is undefined are points of the form $l \cap e$, where l is a line of L and e is a line extending an edge of the Penrose kite. One can check easily, given Equation 1, that all such points have first coordinates of the form $m + An$, where $m, n \in \boldsymbol{Z}$. On the other hand, no point of $T(\boldsymbol{Z}^2) \times \{\pm 1\}$ has this form, and no iterate of such a point has this form. ♠

Let Υ denote the square of the outer billiards map. Let $\mathcal{C} = T(\boldsymbol{Z}^2) \times \boldsymbol{Z}'$, where $\boldsymbol{Z}'$ is the set of odd integers. The vector $\Upsilon(x) - x$ is always twice the difference between two of the vertices of our shape S. Given Equation 1, we see that the first coordinate of $\Upsilon(x) - x$ always has the form $2m + 2nA$ where m and n are integers. The second coordinate is always an even integer. Hence Υ is entirely defined on $\mathcal{C}$ and preserves this set.

Let $\mathcal{C}(+)$ denote the set of points in $\mathcal{C}$ whose second coordinate is 1 and whose first coordinate is positive. Let $\mathcal{C}(-)$ denote the same set, but with the second coordinate being -1. Let $\mathcal{C}(\pm) = \mathcal{C}(+) \cup \mathcal{C}(-)$. This set is dense in the union of two rays starting at $(0,\pm 1)$ and parallel to $(1,0)$. Clearly Υ does not preserve $\mathcal{C}(\pm)$. However, it makes sense to speak of the first return map

$$\Upsilon_R : \mathcal{C}(\pm) \to \mathcal{C}(\pm). \tag{4}$$

Given $x_1, x_2 \in \boldsymbol{Z}^2$ we write $x_1 \leftrightarrow x_2$ if there are choices $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ such that

$$\Upsilon_R(T(x_1), \epsilon_1) = (T(x_2), \epsilon_2). \tag{5}$$

Note that the map $(x, y) \to (x, -y)$ conjugates the outer billiards map to its inverse. It then follows from symmetry that $x_1 \leftrightarrow x_2$ if and only if $x_2 \leftrightarrow x_1$.

We let $\Gamma \subset \boldsymbol{R}^2$ denote the graph obtained by joining points x_1 and x_2 by a line segment if and only if $x_1 \leftrightarrow x_2$. We call Γ the *arithmetic graph*. Let Γ_0 denote the component of the arithmetic graph containing $(0, 0)$. Thus, Γ_0 encodes the structure of the return map on the two points $(T(0, 0), \pm 1)$.

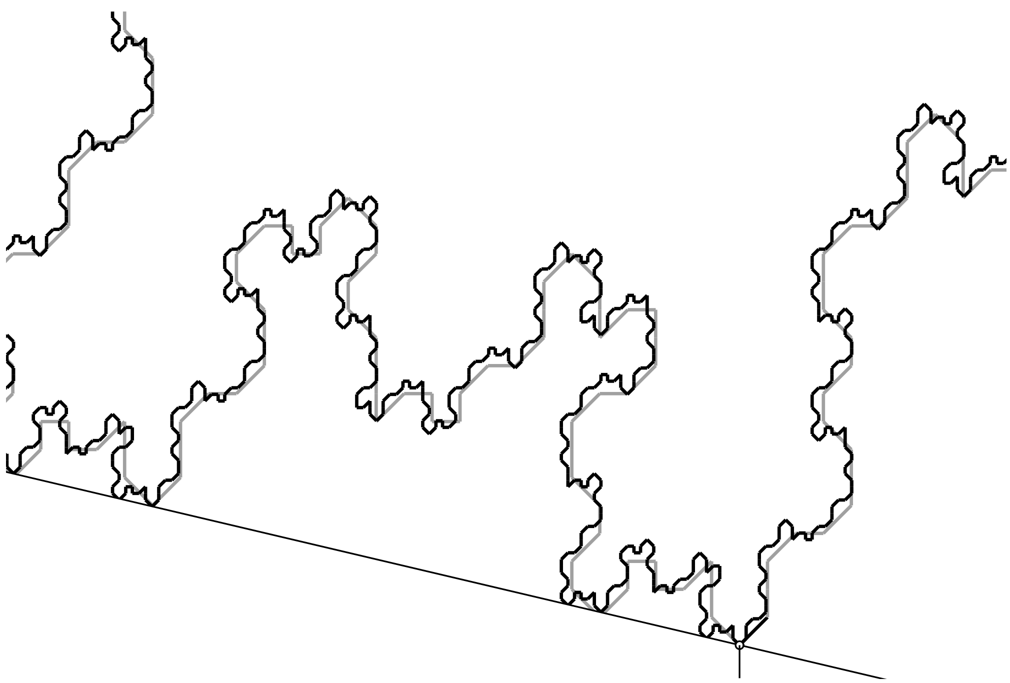

Figure 1: Γ_0 and a dilated copy of Γ_0

Figure 1 shows some of Γ_0, drawn in black. The origin is denoted by a little vertical line segment (nearly) touching ∂H at its top endpoint. Note that $(0, 0)$ lies very slightly above the black line, but this is hard to tell from the picture. (See Billiard King for great pictures.) There is a second curve in Figure 1. We have dilated Γ_0 by ϕ^3 about the origin, drawn it grey, and superimposed it on Γ_0. It seems that Γ_0 and $\phi^3\Gamma_0$ lie within small tubular neighborhoods of each other. (One might say that Γ_0 is *quasi-self-similar*.) We prove this result in [**S**], and deduce that both ends of Γ_0 must exit every tubular neighborhood of ∂H. Our main result follows immediately.

In the next chapter we will give a recipe for drawing the arithmetic graph. We end this chapter with a few remarks on our proof that Γ_0 really is quasi-self-similar. It turns out that there is a partition $\mathcal{P}$ of the square torus T^2 into 26 convex polygons. There is also a *classifying map* $\Psi : \boldsymbol{Z}^2 \to T^2$, given by

$$\Psi = \psi \circ T; \qquad \psi(x) = [\frac{x}{2\phi}, \frac{x}{2}] \tag{6}$$

Here ϕ is the golden mean, and $[(x,y)]$ is the projection of $(x,y) \in \boldsymbol{R}^2$ to T^2, and T is the map from Equation 2. Of course, the map ψ depends on our specific choice of $A = \text{dec}(\sqrt{5}) = \phi^{-3}$.

Ψ has the following nice property. For any $(x,y) \in \boldsymbol{Z}^2$, the local picture of Γ around (x,y), meaning that the edges of Γ incident to (x,y), are determined by which polygon of $\mathcal{P}$ contains $\Psi(x,y)$. Presumably there is something like this for other parameters of A in Equation 1.

Associated to $\mathcal{P}$ is a certain polygon exchange map, and one can produce finer partitions of T^2 by pulling $\mathcal{P}$ back using the dynamics. These finer partitions classify longer pieces of Γ. Say that a *strand* of Γ is a finite polygonal arc of Γ. Say that a *strand-type* is a translation equivalence class of strands. Then we can associate to each strand type of Γ a convex polygon in R^2 that classifies it: $\Psi(x,y)$ lands in this polygon if and only if the local picture of Γ around (x,y) matches the strand type.

There is a certain family of contractions defined on simply connected subsets of T^2. Such contractions have contraction factor ϕ^{-3}, and are intertwined by Ψ with the dilation map discussed above. That is,

$$\Psi(\Phi(x,y)) = \gamma(\Psi(x,y)) \tag{7}$$

where $\Phi(x,y)$ is obtained by choosing an appropriate lattice point near $(\phi^3 x, \phi^3 y)$ and γ is one of the contractions in our family.

Establishing the quasi-self-similarity amounts to showing containments of the form

$$\gamma(P) \subset P' \tag{8}$$

where P is a polygon that classifies a short strand-type of Γ and P' is a polygon that classifies the longer strand-type of Γ that supposedly shadows the dilated version of the shorter strand.

In the end we have to make 75 checks like this, and we do the calculations on the computer, using exact integer arithmetic. We refer the reader to [**S**] for the details.

3 Drawing the Arithmetic Graph

First we describe a factorization of the return map Υ_R into 8 simpler maps, which we call *strip maps*.

Say that a *strip* is a region $S \subset \boldsymbol{R}^2$ bounded by 2 parallel lines, $\partial_0 S$ and $\partial_1 S$. Let V be a vector such that $\partial_0 S + V = \partial_1 S$. That is, V points from one boundary component to the other. Given the pair (S, V) we (generically) define a map $E : \boldsymbol{R}^2 \to S$ as follows. For each generic $x \in \boldsymbol{R}^2$ we define $E(x) = x - nV$ where n is the unique integer such that $E(x) \in S$. This map is well defined unless x lies in a discrete infinite family of parallel lines.

There is a unique affine functional $f(x,y) = a_1 x + a_2 y + a_3$ such that $f_L(V) = 1$, and $f(x,y) \in (0,1)$ iff $(x,y) \in S$. Here $f_L(x,y) = a_1 x + a_2 y$ is the linear part of f. Given f we have the following explicit formula:

$$E(p) = p - \text{floor}(f(p))\ V. \tag{9}$$

Equation 9 is defined unless $f(p)$ is an integer. The triple $\alpha = (a_1, a_2, a_3)$ characterizes f.

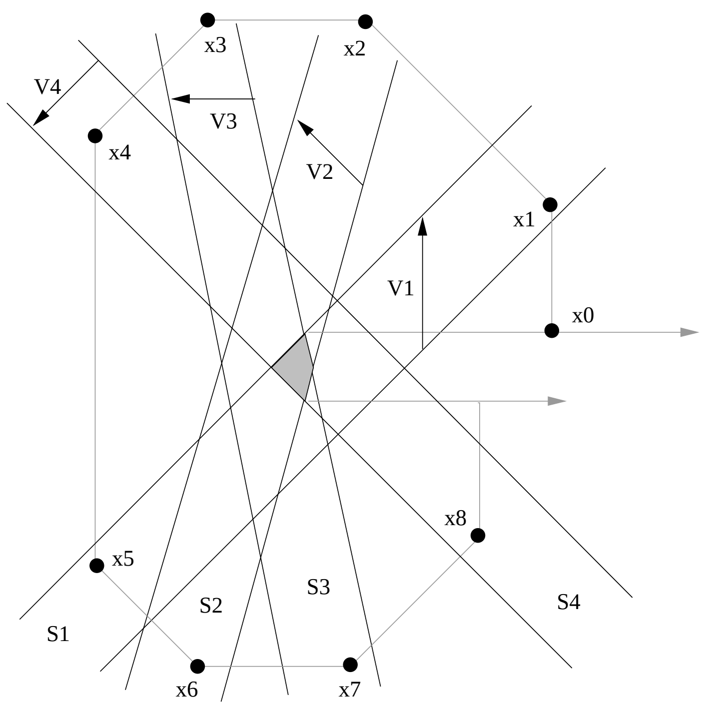

Figure 2: Strip Decomposition of the Return Map

Now for the main construction. Each strip in Figure 2 is obtained by extending an edge of the side of S, and then rotating this extended side through the off-diagonal vertex (top or bottom) that does not contain the edge. We call the strips S_1, S_2, S_3, S_4. We set $E_j = (S_j, V_j)$ for $j = 1, 2, 3, 4$ and $E_{j+4} = E_j$. Here are the triples and the vectors associated to these edge maps.

$$\begin{array}{ll} \alpha_1 = (-1/4, +1/4, +3/4) & V_1 = (0, 4) \\ \alpha_2 = (-\phi/4, +1/2 - \phi/4, +1/2 - \phi/4) & V_2 = (-2, +2) \\ \alpha_3 = (-\phi/4, -1/2 - \phi/4, +1/2 - \phi/4) & V_3 = (+4 - 4\phi, 0). \\ \alpha_4 = (-1/4, -1/4, +3/4) & V_4 = (-2, -2). \end{array}$$

Given a point $x_1 \in \mathcal{C}(\pm)$, at least (say) 10 units from the origin, we can define the points $x_2, ..., x_8$ inductively, like this: $j = 2, ..., 8$ let $x_j = E_j(x_{j-1})$. In [**S**] we prove the following result, which is probably quite well known.

Lemma 3.1 (Pinwheel) *For any $x_0 \in \mathcal{C}(\pm)$ the points x_8 and $\Upsilon_R(x_0)$ lie on the same vertical line.*

Given the Pinwheel Lemma, we can compute the arithmetic graph just by iterating the maps $E_1, ..., E_8$ and checking the results. Here is the algorithm that produces the 1-neighborhood of Γ about (a, b):

- Start with $(a, b) \in \boldsymbol{Z}^2$ such that $x = T(a, b) > 0$.
- Set $x_0 = (x, 1)$ and form the polygon $x_0, ..., x_8$ shown in Figure 2.
- Define numbers $n_1, ..., n_8$ by the rule that the segment connecting x_{k-1} to x_k is n_k times as long as the vector V_2.
- Let $m_k = n_k - n_{k+4}$.
- The first coordinate of $\Upsilon_1(x_0) - x_0$ is then $2Am_3 + 2(m_2 + m_3 + m_4)$.
- Therefore connect (a, b) to $(a, b) + (m_3, m_2 + m_3 + m_4)$. This edge is part of the arithmetic graph.
- Repeat the above steps using $(x, -1)$ in place of $(x, 1)$.

If you apply this algorithm for an entire block of points in $\boldsymbol{Z}^2$, you can see as much of Γ as you care to see. Billiard King implements this algorithm.

4 References

[**B**] P. Boyland, *Dual Billiards, twist maps, and impact oscillators*, Nonlinearity **9** (1996) 1411-1438

[**D**], R. Douady, *These de 3-eme cycle*, Universite de Paris 7, 1982

[**DT**] F. Dogru and S. Tabachnikov, *Dual Billiards*, Math Intelligencer vol. 27 No. 4 (2005) 18–25

[**G**] D. Genin, *Regular and Chaotic Dynamics of Outer Billiards*, Penn State Ph.D. thesis (2005)

[**GS**] E. Gutkin and N. Simanyi, *Dual polygonal billiard and necklace dynamics*, Comm. Math. Phys. **143** (1991) 431–450

[**Ke**] R. Kenyon, *Inflationary tilings with a similarity structure*, Comment. Math. Helv. **69** (1994) 169–198

[**Ko**] Kolodziej, *The antibilliard outside a polygon*, Bull. Polish Acad Sci. Math. **37** (1989) 163–168

[**M**] J. Moser, *Stable and Random Motions in Dynamical Systems, with Special Emphasis on Celestial Mechanics*, Annals of Math Studies **77**, Princeton University Press (1973)

[**N**] B.H. Neumann, *Sharing Ham and Eggs*,
summary of a Manchester Mathematics Colloquium, 25 Jan 1959
published in Iota, the Manchester University Mathematics students' journal

[**S**] R. E. Schwartz, *Unbounded Orbits for Outer Billiards*, Journal of Modern Dynamics **3** (2007) to appear

[**T**] S. Tabachnikov, *Geometry and Billiards*, A.M.S. Mathematics Advanced Study Semesters (2005)

[**VS**] F. Vivaldi, A. Shaidenko, *Global stability of a class of discontinuous dual billiards*, Comm. Math. Phys. **110** (1987) 625–640