An Improved Bound on the Optimal Paper Moebius Band: Erratum

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This erratum concerns a mistake I made in my paper *An Improved Bound for the Optimal Paper Moebius Band*. While the mistake is rather embarrassing, I discovered that when I fixed the mistake the resulting calculation solved the whole conjecture. I will explain the mistake, explain how it impacts the paper, then do the calculation which solves the conjecture.

I have written a second paper which gathers together the corrected calculation, as well as an improved proof of the main result in this paper, and gives a nicely written solution to the whole conjecture. The new paper is entitled *The Optimal Paper Moebius Band* and I posted it on the arXiv.

Consider the space

$$M_\lambda = ([0,1] \times [0,\lambda])/\sim, \qquad (x,0) \sim (1-x,\lambda) \tag{1}$$

Suppose B is a line segment in M_λ which has both endpoints in the boundary and otherwise lies in the interior. If you cut M_λ open along B you get a trapezoid. In my paper I claimed that you get a parallelogram.

This mistake does not affect the main result in the paper, the lemma about the existence of T-patterns. It also does not affect the main point of the paper which is to show that there is a number $\lambda_1 > \pi/2$ which bounds the aspect ratio of a paper Moebius band. The one-paragraph argument I give showing that $\lambda_1 \geq (1+\sqrt{5})/2$ works just fine.

The mistake does affect the optimization problem I set up in §3 of the paper, the one which calculates the bound I get in the Main Theorem. In

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particular, the left hand side of Figure 3.1 should be a trapezoid and not a rhombus. When the calculation is redone correctly, the new bound is $\sqrt{3}$, and this solves the optimal paper Moebius band conjecture.

Here is the corrected calculation. Figure 3.1' is like Figure 3.1 in the paper except that the figure on the left is a trapezoid and the labels are different. On the left we have M_λ cut open along B. The segment T is also a straight line segment. The top and bottom of this trapezoid are meant to be glued to make a Moebius band. The segments B' and T' are images of B and T respectively under an isometric embedding $I : M_\lambda \to M \subset \boldsymbol{R}^3$. These segments are disjoint, perpendicular, and co-planar. The second B lies below the line extending T'.

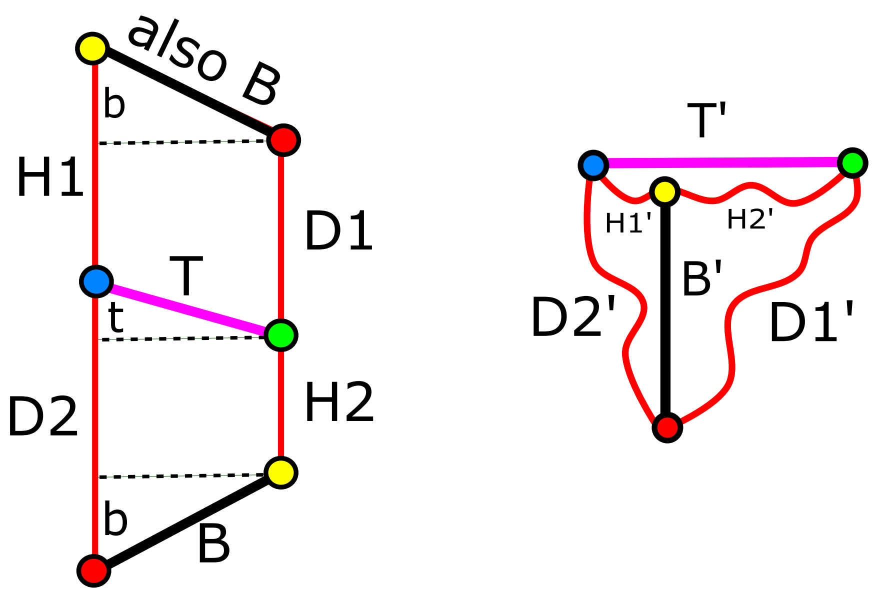

Figure 3.1': The corrected diagram

I've drawn one of 4 basic pictures you could draw. In some of the other pictures, B slants downward on the bottom and upward at the top. In some of the other pictures, T slants upwards. the calculations work out the same way for all possibilities. The formulas are all the same. This one is easiest on the eyes when you want to calculate things. We define $(-t)$ to be the slope of T and b to be the slope of the bottom choice of B, We have

$$\ell(D_2) = \ell(H_2) + t + b, \qquad \ell(D_1) = \ell(H_1) + t - b. \tag{2}$$

$$\ell(T') = \ell(T) = \sqrt{1+t^2}. \tag{3}$$

It turns out that the perimeter of a triangle having base $\sqrt{1+t^2}$ and height at least 1 is at least

$$P(t) = \sqrt{1+t^2} + \sqrt{5+t^2}. \tag{4}$$

Since $I(\partial M_\lambda)$ contains all vertices of Δ, we have

$$2\lambda \geq P(t). \tag{5}$$

Also the sum of the two diagonal sides of Δ is at least $\sqrt{5+t^2}$. The length of the boundary of the Moebius band is

$$\ell(D_1) + \ell(D_2) + \ell(H_2) + \ell(H_2) = 2\ell(D_1) + 2\ell(D_2) - 2t. \tag{6}$$

As the same time the combined length of D_1 and D_2 is at least the length of the non-diagonal sides. Putting everything together,

$$2\lambda \geq 2\sqrt{5+t^2} - 2t. \tag{7}$$

Now we can say that

$$2\lambda \geq \max\left(2\sqrt{5+t^2} - 2t, \sqrt{1+t^2} + \sqrt{5+t^2}\right). \tag{8}$$

It an easy exercise to show that the right side equation exceeds $2\sqrt{3}$, with equality if and only if $t = 1/\sqrt{3}$. Hence $\lambda \geq 1/\sqrt{3}$.

This is the bound conjectured by Halpern and Weaver in 1977.