

Patterns of Geodesics, Shearing, and Anosov Representations of the Modular Group

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Abstract

Let $X = SL_3(\mathbf{R})/SO(3)$. Let \mathcal{DFR} be the space of discrete faithful representations of the modular group into $\text{Isom}(X)$ which map the order 2 generator to an isometry with a unique fixed point. This paper continues my work on the Barbot component \mathcal{B} of \mathcal{DFR} , showing how all the representations in \mathcal{B} are shears of the Pappus modular groups in an appropriate sense.

1 Introduction

Let $X = SL_3(\mathbf{R})/SO(3)$. This is a prototypical higher rank symmetric space. In this paper we consider the moduli space \mathcal{DFR} of conjugacy classes of discrete and faithful representations of the modular group $\mathbf{Z}/3 * \mathbf{Z}/2$ into $\text{Isom}(X)$ which map the order 2 elements to isometries having a unique fixed point in X .

The *Pappus representations* are a 2-parameter subfamily of \mathcal{DFR} which I constructed in my 1993 paper [S0] and then revisited in my recent paper [S1]. These groups exhibit many features that, much later and more generally, appeared in higher Teichmüller Theory, e.g. in [Lab], [GW], [Bar], [BCLS], and [KL].

The Pappus representations arise as the projective symmetry groups of *convex marked box orbits*. A *convex marked box* is convex quadrilateral with

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two additional marked points on a pair of opposite edges. One gets a marked box orbit by starting with one marked box and iteratively applying operations that are derived from Pappus's Theorem, a classic theorem in projective geometry.

The Pappus representations are nowadays classified as *relatively Anosov groups in the Barbot component*. This point of view is exposed in [BLV] and [KL]. Let $\mathcal{P} \subset \mathcal{DFR}$ denote the subset consisting of Pappus modular group representations. The component \mathcal{B} of \mathcal{DFR} containing \mathcal{P} is called the *Barbot component*. It is partially understood thanks to [S0] and [BLV].

In [BLV], T. Barbot, G.-S. Lee, and V. P. Valerio build on [S0] and construct a 3-parameter family of Anosov representations which are defined in terms of modified operations on marked boxes. Using their *morphed marked boxes* (my terminology) they construct a 4-parameter family of representations of $\mathbf{Z}/3 * \mathbf{Z}/3$ into $SL_3(\mathbf{R})$, all of which are Anosov. Using an Implicit Function argument, they show that a 3-parameter subset of these are index 2 subgroups of Anosov elements of \mathcal{DFR} . I gave ¹ a complete algebraic analysis of their construction in [S2].

Now I will summarize the results of [S2]. Let \mathcal{R} be the space of representations of $\mathbf{Z}/3 * \mathbf{Z}/2$ with the following property: The order 2 generator is mapped to an isometry with a unique fixed point in X , and this point is not contained in the fixed set of the order 3 generator.

Theorem 1.1 *The space \mathcal{R} is homeomorphic to $\mathbf{R}^3 - \{0\}$. The set \mathcal{P} of Pappus representations is a properly embedded plane in \mathcal{R} . The connected component of $\mathcal{R} - \mathcal{P}$ that is disjoint from the origin consists entirely of Anosov representations in \mathcal{DFR} . The union of \mathcal{P} with this Anosov component is a connected component of \mathcal{DFR} .*

Let $\mathcal{B} \subset \mathcal{R}$ be the union of \mathcal{P} and the Anosov component of $\mathcal{R} - \mathcal{P}$. In this paper, I will give alternate descriptions of the groups in \mathcal{B} as isometry groups of certain patterns of geodesics and flats in X . There are two descriptions per representation, distinct in the generic case. I call these the *prism descriptions*. (The end of this introduction explains the name.) The prism descriptions and the marked box description complement each other: Properties of the parametrization, such as injectivity, are easy to see from the

¹The original version of this paper contained this analysis. However, I split the original version of this paper into two halves, offloading the analysis of the Barbot-Lee-Valerio construction to a separate paper. I did this because the original paper was very long.

prism descriptions, and the discreteness/Anosov properties of the individual representations are easier to see from the morphed marked box description. In particular, all the hard work concerning the Anosov nature of the representations is already done in [BLV].

The prism descriptions are an elaboration of my paper [S0], in which I re-interpreted the Pappus modular groups as isometry groups of what I call *Farey patterns*. These are embedded patterns of geodesics in X that have the same asymptotic structure as the Farey triangulation. Each geodesic in the Farey pattern is a *medial geodesic*. These are geodesics which are angle bisectors of the Weyl chambers within the flats that contain them. These patterns do not generally lie in a totally geodesic slice of X . They are bent, sort of like pleated planes.

Theorem 1.2 *Let $\rho \in \mathcal{B} - \mathcal{P}$ be arbitrary. Let $\Gamma = \rho(\mathbf{Z}/3 * \mathbf{Z}/2)$. Then there are two infinite embedded patterns $Y_{\rho,1}$ and $Y_{\rho,2}$ of geodesics in X such that $\Gamma \subset \text{Isom}(Y_{\rho,1})$ and $\Gamma \subset \text{Isom}(Y_{\rho,2})$ with finite index. In the generic case, Γ equals these isometry groups, and $Y_{\rho,1}, Y_{\rho,2}$ are not isometric to each other. Finally, the pattern of flats containing $Y_{\rho,k}$ is embedded for $k = 1, 2$.*

Just as in the classic Farey triangulation, the geodesics in the Farey patterns are organized into triples of mutually asymptotic geodesics which I call *triangles*. (The triple of flats containing the geodesics in a triangle are the *prisms*.) Two triangles τ_1, τ_2 are *adjacent* if they have a geodesic γ in common. The pair (τ_1, τ'_2) is a *shear* of (τ_1, τ_2) if $\tau'_2 = I(\tau_2)$ where I is an isometry of X which translates along γ . There is a 1-parameter family of such translations. The amount of shearing, which we call the *strength* of the shear, is the translation length of I . More generally, if J is an isometry of X , the pair $(J(\tau_1), J(\tau'_2))$ is a shear of (τ_1, τ_2) .

Speaking more globally, a *shear* of a Farey pattern is another pattern of geodesics in which adjacent triangles in the new pattern are shears of adjacent triangles in the original pattern – all of the same strength and in the same direction so to speak.

Theorem 1.3 *The space \mathcal{B} has two different 1-dimensional foliations by proper and canonically parametrized rays. Each ray has its endpoint in \mathcal{P} . A representation $\rho \in \mathcal{B}$ lies d units along a ray in the first foliation if and only if it lies d units along a ray in the second foliation. In this case, the patterns $Y_{\rho,1}$ and $Y_{\rho,2}$ are each strength- d shears respectively of Farey patterns $F_{\rho,1}$ and $F_{\rho,2}$. Generically, $F_{\rho,1}$ and $F_{\rho,2}$ are not isometric to either.*

For each d , the double foliation sets up a map $\phi_d : \mathcal{P} \rightarrow \mathcal{P}$, which I call the *shearing dynamics*. Here is a description. We start with a Pappus representation p_0 , and then move d -units along the ray in the first foliation that contains it. We arrive at some representation p_d . Then we move d units backwards along the ray in the second foliation that contains p_d . This gives us $\phi_d(p)$. The map ϕ_d is a self-homeomorphism of \mathcal{P} that is the product of 2 involutions. It is the *bounce map* associated to the initial segments of length d of our two ray foliations.

I did a little bit of experimentation with these shearing dynamics. (I am not yet completely confident in my way of computing it.) For the parameters I looked it, it seems that ϕ_d has infinite order and that its orbits are dense in finite unions of 1-dimensional curves.

This paper is organized as follows.

- In §2, I discuss classic shearing of the modular group in the hyperbolic plane. The shearing phenomenon we uncover in X extends what happens in \mathbf{H}^2 .
- In §3, I give some background material about the space X .
- In §4, I explore the geometry of *prisms*, namely triples of flats which contain mutually asymptotic medial geodesics. (I only consider prisms defined by flats having negative triple invariants; the other kinds of prisms are presumably related to the Goldman-Hitchin component [**G1**], [**Hit**] of \mathcal{DFR} .) At the end of §4, I describe a space \mathcal{A} of pairs (Π, p) where Π is a prism and $p \in \Pi$ is some point. There is a map $\rho : \mathcal{A} \rightarrow \mathcal{R}$ which creates a modular group representation based on the data (Π, p) .
- In §5, I show that the map ρ is injective on \mathcal{P} and two-to-one on $\mathcal{A} - \mathcal{P}$. More precisely, I consider the two components \mathcal{A}_+ and \mathcal{A}_- of $\mathcal{A} - \mathcal{P}$. I show that ρ is injective on each of \mathcal{A}_+ and \mathcal{A}_- , and I show that

$$\rho(\mathcal{P} \cup \mathcal{A}_+) = \rho(\mathcal{P} \cup \mathcal{A}_-) = \rho(\mathcal{A}).$$

So, in short, ρ is a kind of folding map, just as in the real hyperbolic case. At the end of §5, I do a calculation related to the gradient of the trace function on \mathcal{R} . At this point, I arbitrarily choose \mathcal{A}_+ over \mathcal{A}_- . The calculations in this chapter are done in Mathematica [**W**].

- In §6, I work out the topology of $\mathcal{P} \cup \mathcal{A}_+$ and show that $\rho : \mathcal{P} \cup \mathcal{A}_+ \rightarrow \mathcal{R}$ is continuous, injective, and proper. Combining this with the trace calculation at the end of §5, I conclude that $\rho(\mathcal{A}) = \mathcal{B}$, the component from Theorem 1.1. Hence every group in $\mathcal{A} - \mathcal{P}$ is Anosov and every group in \mathcal{B} is a prism group.
- In §7, I show that the patterns of geodesics in Theorem 1.2 are always embedded, and so are the patterns of flats that contain them.

One can download the Mathematica code for this paper from <http://www.math.brown.edu/~res/PappusCalcs.TAR>

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2 The Classic Case

2.1 The Hyperbolic Plane

We work with the upper half-plane model \mathbf{H}^2 of the hyperbolic plane. In this model, the geodesics are either arcs of semicircles with endpoints on \mathbf{R} or else vertical rays. The group $\text{Isom}(\mathbf{H}^2)$ is generated by real linear fractional transformations and the map $z \rightarrow -\bar{z}$, which is reflection in the Y -axis.

A group $\Lambda \subset \text{Isom}(\mathbf{H}^2)$ acts *discretely* if for any compact $K \subset \mathbf{H}^2$ there are only finitely many $g \in \Lambda$ such that $g(K) \cap K \neq \emptyset$. This kind action is also called *properly discontinuous*. The *limit set* of Λ is the accumulation set on $\mathbf{R} \cup \infty$ of any orbit. The definition does not depend on the orbit chosen.

2.2 The Farey Triangulation

The geodesics of the Farey triangulation limit on rational points in the ideal boundary $\mathbf{R} \cup \infty$; two rationals a/b and c/d are endpoints of a geodesic in the triangulation if and only if $|ad - bc| = 1$. See Figure 2.1

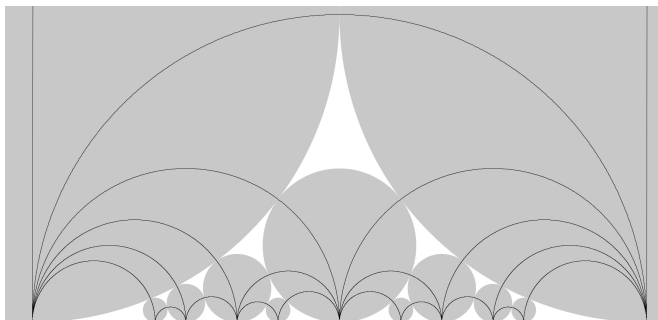


Figure 2.1: Part of the Farey triangulation and dual horodisk packing

The tangency points of the horodisks in the packing are distinguished points on the geodesics of the Farey pattern. We call these the *inflection points* of the geodesic. A more robust definition of the inflection points goes like this: Any ideal triangle in \mathbf{H}^2 has an order 6 symmetry group. The elements of order 2 are reflections about geodesics which connect an inflection point to the opposite cusp. This definition is nicer because it only depends on the individual ideal triangle. When the triangles are arranged as in the Farey triangulation, the robust definition coincides with the special definition given in terms of the horodisks.

2.3 The Modular Group and its Shears

The modular group $PSL_2(\mathbf{Z})$ is generated by the order 3 isometric rotations about the centers of the ideal triangles in the triangulation and the order 2 reflections about the inflection points. Algebraically, the modular group is the free product $\mathbf{Z}/2 * \mathbf{Z}/3$.

The robust definition of the inflection points gives us another way to define the modular group. Let τ be some fixed ideal triangle. Then the modular group is the isometry group generated by the order 3 counterclockwise rotation σ_3 of τ and by the order 2 rotation σ_2 about one of the inflection points of τ . If we choose τ to be (say) the triangle with vertices $0, 1, \infty$ then we recover the modular group exactly. If we start with a different choice of τ we get a group that is conjugate to the modular group.

Now we consider shearing of the modular group. Let τ be an ideal triangle, as above. Let γ be one of the geodesics comprising $\partial\tau$. We choose one of the points $p \in \gamma$ which is s units from the inflection point on γ . We then let Γ_s denote the group generated by σ_3 above and by the order 2 rotation

about p . When $s = 0$ we recover the modular group. When $s > 0$ we get a *shearing* of the modular group. The other choice of $p \in \gamma$ that is equidistant from the inflection point gives a conjugate group. So, the distance s here is all that really matters.

When $s \neq 0$, the group Γ_s preserves a tiling of a closed subset $\Delta_s \subset \mathbf{H}^2$ by ideal triangles. We get this tiling starting with τ and using the isometries to successively lay down isometric copies of τ . Two adjacent ideal triangles τ_1 and τ_2 are related in the following way. Let γ be the geodesic common to τ_1 and τ_2 . Then the distance between the inflection point of τ_1 on γ and the inflection point of τ_2 on γ is $2s$. Thus, we can also think of getting the pair (τ_1, τ_2) by starting with two adjacent triangles in the Farey triangle, sliding one of them $2s$ units relative to the other, then moving the union into some new position by an isometry.

The limit set Λ_s of Γ_s is a Cantor set when $s \neq 0$. The region Δ_s is the convex hull of Λ_s . The group Γ_s is a classic example of an Anosov group. As $s \rightarrow 0$, the region Δ_s converges to all of \mathbf{H}^2 (assuming we keep the initial triangle τ the same for all s) and the limit set Λ_s converges to $\mathbf{R} \cup \infty$ in the Hausdorff topology.

The description above is quite well known. See e.g. [T] or [P].

2.4 The Representation Variety

To get a representation of the abstract modular group $\mathbf{Z}/2 * \mathbf{Z}/3$ all we need to do is choose an order 2 element and an order 3 element. We will insist that our representation is in $PSL_2(\mathbf{R})$, the index 2 subgroup of linear fractional transformations. We consider two representations equivalent if they are conjugate. Once we do this, the only data that is important is the distance d between the fixed points. This means that the representation variety is just the ray $R = [0, \infty)$. The point $0 \in R$ corresponds to the case when both fixed points coincide.

Let d_0 denote the distance between the center of an ideal triangle and its inflection point. Our distance parameter d in this section is related to the shear parameter s from the previous section by a formula one can work out with hyperbolic trigonometry. The case $d = d_0$ corresponds to $s = 0$ and the case $d > 0$ corresponds to $s \neq 0$. As is well known, the point $d \in R$ gives rise to a discrete and faithful (i.e. injective) representation if and only if $d \geq d_0$.

Here is another way to describe the trichotomy. Let

$$g = \sigma_2 \circ \sigma_3. \tag{1}$$

Then g^2 is elliptic, parabolic, or loxodromic according as $d < d_0$, $d = d_0$, or $d > d_0$. Since g^2 is given by a linear fractional transformation based on an element of $SL_2(\mathbf{R})$, this criterion can be expressed in terms of $\text{Trace}(g^2)$ being either less than, equal to, or greater than 2.

Let us reconcile this with our picture of the shears. The shears of the modular group give what looks like a 1-parameter family of representations that is diffeomorphic to \mathbf{R} . After all, we are free to slide the point anywhere along the geodesic γ . However, this copy of \mathbf{R} maps into R with a fold at d_0 : The image is the ray $[d_0, \infty)$. If we shear the same amount in opposite directions, we get conjugate groups. In particular, every group aside from the modular group has 2 distinct descriptions in terms of shearing. This kind of folding picture will generalize to the case of $X = SL_3(\mathbf{R})/SO(3)$.

3 Geometric Preliminaries

3.1 The Symmetric Space

Here I give an abbreviated account of the corresponding material in [S1]. This material is, of course, well known.

Basic Definition: The symmetric space $X = SL_3(\mathbf{R})/SO(3)$ can be interpreted as the space of unit volume ellipsoids centered at the origin of \mathbf{R}^3 . There is a natural origin of X , the point which names the round ball. The group $SL_3(\mathbf{R})$ acts on X in the obvious way. If E is an ellipsoid and $T \in SL_3(\mathbf{R})$ then $T(E)$ is just the image ellipsoid. Here I am somewhat blurring the distinction between points in X and the ellipsoids they name. The group $SL_3(\mathbf{R})$ acts transitively on X and the stabilizer of the origin is $SO(3)$. So, the orbit map gives an isomorphism between the coset description of X and the ellipsoid description.

One can also interpret X as the space of unit determinant positive definite symmetric matrices. Each matrix like this defines an inner product on \mathbf{R}^3 and this unit ball of this inner product is a unit volume ellipsoid centered at the origin. This is how the correspondence between two descriptions works.

The Metric: The space X has a canonical $SL_3(\mathbf{R})$ invariant metric which is induced by a Riemannian metric of non-positive sectional curvature. The distance between E_0 and the *standard ellipsoid* $E(a, b, c)$ given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a, b, c > 0, \quad abc = 1. \quad (2)$$

is

$$\|(\log(a), \log(b), \log(c))\| = \sqrt{\log^2(a) + \log^2(b) + \log^2(c)}. \quad (3)$$

The rest of the metric can be deduced from symmetry.

Isometries: As already mentioned, $SL_3(\mathbf{R})$ acts isometrically on X . There is also an order 2 isometry Δ of X which fixes the origin and reverses all the geodesics through the origin. In terms of the matrix interpretation of X , this isometry is given by $S \rightarrow S^{-1}$, where S is a positive definite symmetric matrix. We call this map the *standard polarity*, for reasons discussed in §3.2. The standard polarity maps the ellipsoid $E(a, b, c)$ to $E(1/a, 1/b, 1/c)$.

The group $\text{Isom}(X)$ is generated by Δ and $SL_3(\mathbf{R})$. We call the elements of $SL_3(\mathbf{R})$ *projective transformations* and the remaining elements *dualities*. In particular, any point of X is fixed by an order 2 isometry (a conjugate of Δ) which reverses all the geodesics through that point. We will call such an isometry an *elliptic polarity*. These names are usually reserved for maps on the projective plane and its dual, but here we are talking about the action on X . We explain the connection in §3.2.

Flats: The *standard flat* F_0 is the union of all the points representing standard ellipsoids. The rank 2 abelian group of diagonal matrices acts transitively on the standard flat. Thus, F_0 is isometric to a Euclidean plane. In particular, the straight lines in F_0 are geodesics in X . Every other flat in X is isometric to F_0 . In particular, the structure of F_0 determines the structure of all the flats.

There are 3 *singular geodesics* through the origin in F_0 : These correspond to the standard ellipsoids where the set $\{a, b, c\}$ has cardinality at most 2. That is, either $a = b$ or $a = c$ or $b = c$. In general, the *singular geodesics* in F_0 are the ones parallel to the singular geodesics through the origin. A geodesic in F_0 is contained in more than one flat if and only if it is a singular geodesic. All other geodesics in F_0 lie only in F_0 .

There are 3 *medial geodesics* through the origin. These correspond to triples (a, b, c) where either $a = 1$ or $b = 1$ or $c = 1$. More generally, a

medial geodesic in F_0 is one parallel to a medial geodesic through the origin. Each medial geodesic lies in a unique flat. In terms of the cyclic order on the geodesics through the origin in F_0 , the singular geodesics alternate with the medial geodesics, and the angle between adjacent singular and medial geodesic is $\pi/6$. A *medial foliation* of a flat is a foliation by parallel medial geodesics. Thus, every flat has 3 medial foliations. We call a flat with a distinguished medial foliation a *marked flat*.

Visual Boundary: The visual boundary of X is defined to be the union of geodesic rays through the origin. We denote this as ∂X . The action of isometries on X extends to give a homeomorphism of ∂X in the following way. If ρ is a geodesic ray through the origin and I is an isometry then the image of ρ under I is some other geodesic ray, not necessarily contained on a geodesic through the origin. There is a unique geodesic ray through origin ρ' such that the distance between corresponding points of $I(\rho)$ and ρ' remains uniformly bounded. The action of I on ∂X maps ρ to ρ' .

3.2 Connection to Projective Geometry

Projective Objects: The projective plane \mathbf{P} is the set of 1-dimensional subspaces of \mathbf{R}^3 . The dual plane \mathbf{P}^* is the set of 2-dimensional subspaces of \mathbf{R}^3 . The *flag variety* is the set of pairs (p, ℓ) where p is a 1-dimensional subspace of \mathbf{R}^3 and ℓ is a 2-dimensional subspace of \mathbf{R}^3 , and $p \subset \ell$. These objects are called *flags*. Equivalently, a flag is a pair (p, ℓ) where p is a point of \mathbf{P} and ℓ is a line of \mathbf{P} , and $p \in \ell$. Each point in \mathbf{P} corresponds to a line in \mathbf{P} , namely the set of 1-dimensional subspaces contained in a given 2-dimensional subspace.

Limits of Singular and Medial Geodesics: The singular geodesics accumulate at one end to points of \mathbf{P} and at the other end to points of \mathbf{P}^* . This is easily seen for the standard flat. For one of the singular geodesics through the origin, the corresponding standard ellipsoids are $E(a, a, 1/a^2)$. As $a \rightarrow 0$ these become long and thin and pick out a 1-dimensional subspace in \mathbf{R}^3 . As $a \rightarrow \infty$, these ellipsoids flatten out like a pancake and define a 2 dimensional subspace of \mathbf{R}^3 .

The medial geodesics accumulate at both end at points of the flag variety. The standard example is the medial geodesic consisting of $E(a, 1, 1/a)$. As $a \rightarrow 0$ the 1-dimensional subspace is the Z -axis and the 2-dimensional sub-

space is the YZ -plane. As $a \rightarrow \infty$ the 1-dimensional subspace is the X -axis and the 2-dimensional subspace is the XY -plane. The intuition here is that in either direction these ellipsoids look like popsicle sticks. The longest direction picks out the one dimensional subspace and the two longest directions pick out the two dimensional subspace.

Marked Flats and Pairs of Flags: A triple of points in \mathbf{P} is in *general position* if they are not contained in the same line. Likewise, a triple of lines in \mathbf{P} is in general position if they do not have a single point in common. Two flags (p_1, ℓ_1) and (p_2, ℓ_2) are *transverse* if $p_1 \notin \ell_2$ and $p_2 \notin \ell_1$.

A marked flat defines a pair of transverse flags, namely the (common) limits of the medial geodesics in the foliation. Conversely a pair of transverse flags determines a unique marked flat. The space of marked flats is 6-dimensional. A *projective triangle*, namely a triple of general position points, determines a unique flat, and *vice versa*. A pair of transverse flags determines a unique projective triangle, and a projective triangle determines three pairs of transverse flags, corresponding to the three markings of the flat.

Our Terminology Explained: Consider an element of $\text{Isom}(X)$ which we are calling a projective transformation. Such an element acts on ∂X in such a way as to preserve \mathbf{P} . The action the map induces on \mathbf{P} is precisely that of a classical projective transformation: It is a homeomorphism of \mathbf{P} which maps lines to lines.

Consider an element of $\text{Isom}(X)$ which we are calling a duality. Such an element acts on ∂X in such a way as to swap \mathbf{P} and \mathbf{P}^* . The induced map we get on $\mathbf{P} \cup \mathbf{P}^*$ is a duality in the classic sense: It swaps points and lines in such a way as to carry collinear lines to coincident lines.

In particular, the isometries of X we are calling *elliptic polarities* induce the maps on $\mathbf{P} \cup \mathbf{P}^*$ which are commonly called elliptic polarities. Such maps are given by $\ell \leftrightarrow \ell^*$ where the 1-dimensional subspace in \mathbf{R}^3 representing ℓ is mapped to the 2-dimensional subspace representing ℓ^* which is orthogonal to it with respect to some inner product. When this inner product is the usual dot product, we get the standard polarity.

3.3 Matrix Actions

Here we explain how we compute the action of projective transformations and dualities using matrices.

Representing Points and Lines: We represent points in \mathbf{P} as 3-vectors. When $c \neq 0$, the vector (a, b, c) represents the point $(a/c, b/c)$ in the *affine patch*. The affine patch is essentially a copy of \mathbf{R}^2 sitting inside \mathbf{P} . We also represent lines as vectors. The vector (a, b, c) represents the line given by the subspace $ax + by + cz = 0$. If we have two vectors $v_1 = (a_1, b_1, 1)$ and $v_2 = (a_2, b_2, 1)$ then the vector $(1+t)v_1 - tv_2$ represents a point on the line $\overline{v_1 v_2}$.

Action on Points: We will work with matrices in $GL_3(\mathbf{R})$. For our purposes we do not need to fuss about whether our matrix has determinant 1. We can always scale the matrix to have this property. (Here we are strongly using the property that we have $n \times n$ matrices and $n = 3$ is odd; in the even case the situation is a bit different.) The matrix S acts on a vector \hat{p} representing a point p by linear transformation: The new vector $S(\hat{p})$ represents $S(p)$.

Action on Lines: The matrix S acts on our line representations in the following way: We let $(S^{-1})^t$ act on the vector representation \hat{L} of a line L . and the new vector represents the line $S(L)$. Here we are taking the inverse-transpose. A few calculations will convince the reader that this is indeed the right thing to do. This works because

$$S(\hat{p}) \cdot (S^{-1})^t(\hat{L}) = S^{-1}S(\hat{p}) \cdot \hat{L} = \hat{p} \cdot \hat{L}.$$

This way of defining the action is correct because it preserves incidences between points and lines.

Dualities: The standard polarity Δ just acts as the identity matrix, both on points and lines. Thus $\Delta(a, b, c) = (a, b, c)$. All that changes is the interpretation of the meaning of the vector (a, b, c) . It is most convenient to represent dualities as compositions $\Delta \circ S$. We have the equations

$$S \circ \Delta = \Delta \circ (S^{-1})^t, \quad S \circ \Delta \circ S^{-1} = \Delta \circ (S^{-1})^t S^{-1}. \quad (4)$$

The first of these equations implies the second one.

The Representation Space: The modular group $G = \mathbf{Z}/2 * \mathbf{Z}/3$ is generated by σ_2 and σ_3 , elements of order 2 and 3. We consider representations of G into $\text{Isom}(X)$ such that

- $\rho(\sigma_2)$ is an elliptic polarity.
- $\rho(\sigma_3)$ is the matrix given by

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \theta = 2\pi/3. \quad (5)$$

This matrix acts on \mathbf{P} so as to preserve the affine patch and act there as an order 3 rotation about the origin.

- The fixed point of $\rho(\sigma_2)$ does not lie in the fixed point set of $\rho(\sigma_3)$.

The fixed point set of $\rho(\sigma_3)$ is the singular geodesic γ , consisting of standard ellipsoids $E(a, a, a^{-2})$. We call these representations *normalized*. We consider two representations to be the same if they are conjugate in $\text{Isom}(X)$.

Let \mathcal{R} denote the space of all normalized representations, modulo conjugacy. The distance between two elements $[\rho_1], [\rho_2] \in \mathcal{R}$ to be the minimal D such that there are two normalized representatives ρ_1 and ρ_2 such that the fixed point sets of $\rho_1(\sigma_1)$ and $\rho_2(\sigma_2)$ are D apart in X . In [S2] we proved the following result.

Theorem 3.1 *\mathcal{R} is homeomorphic to $\mathbf{R}^3 - \{(0, 0, 0)\}$ and is a smooth manifold away from the two curves, one corresponding to line-preserving representations and one corresponding to conic-preserving representations. The trace of any word is a smooth function on the smooth points of \mathcal{R} .*

4 The Prism Representations

4.1 Basic Definitions

We will study triples of flags $\{(p_i, \ell_i)\}$. We insist that $p_i \notin \ell_j$ when $i \neq j$. Let P_1, P_2, P_3 and L_1, L_2, L_3 respectively be vectors representing p_1, p_2, p_3 and ℓ_1, ℓ_2, ℓ_3 . These vector representatives are unique up to scaling. Our condition is that $P_i \cdot L_j \neq 0$ when $i \neq j$. The *triple product* of our flag triple is

$$\chi = \frac{(P_1 \cdot L_2)(P_2 \cdot L_3)(P_3 \cdot L_1)}{(P_2 \cdot L_1)(P_3 \cdot L_2)(P_1 \cdot L_3)}. \quad (6)$$

This is a very well known invariant. Compare [FG].

We call the triple of flags *negative* if $\chi < 0$. We can always normalize by a projective transformation so that our triple of flags is rotationally symmetric. Figure 4.1 shows the 4 possibilities. In the first two cases the points are not visible because they lie in the line at infinity.

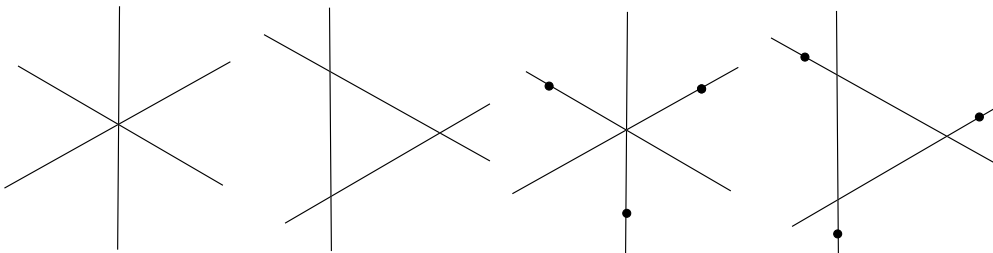


Figure 4.1: Negative triples with 3-fold Euclidean symmetry.

The figure on the right is meant to represent an infinite family of examples. In the figure on the right, we are allowed to move the points in and out in a symmetric way as long as we do not move them into the central triangle. It is worth remarking, finally, that the middle two cases are dual to each other.

We call the triple of flags *generic* if it is projectively equivalent to a member of the family represented by the figure on the right in Figure 4.1.

If we permute the order of our flags then ξ is either preserved or replaced by $1/\xi$. For non-generic negative triples we have $\xi = -1$ and the invariant cannot tell apart the various cases. For the generic case, two negative triples are projectively equivalent if and only if they have the same triple invariant. Referring to Figure 4.1, the invariant χ , when not equal to -1 , measures how the triple of points is placed with respect to the inner equilateral triangle.

Definition: A *prism* is the triple of marked flats corresponding to a negative triple of flags. We define the *triple invariant* of the prism Π to be

$$\xi(\Pi) = |\log(-\chi)| \in [0, \infty), \quad (7)$$

where χ is the triple invariant of a triple of flags defining Π . We take absolute values so as to get an invariant that is independent of permutations of the flags and also self-dual. By construction, the prism Π is generic if and only if $\xi(\Pi) > 0$.

Lemma 4.1 *The generic prisms Π_1 and Π_2 are isometric iff $\xi(\Pi_1) = \xi(\Pi_2)$.*

Proof: Suppose Π_1, Π_2 are generic prisms and $I : P_1 \rightarrow P_2$ is an isometry. Then I maps the one triple of flags to the other and either preserves the triple product or inverts it (depending on whether I comes from a projective transformation or a duality.) Conversely, any two triples with the same or reciprocal triple invariants are equivalent under some isometry of X . ♠

4.2 Inflection Points and Lines

In this section we pick out some special geometric features of prisms, which we call inflection points and inflection lines. The inflection points only exist for the generic prism and the inflection lines exist in all cases.

Lemma 4.2 *The symmetry group of a generic prism is isomorphic to S_3 , the permutation group of order 6. The even permutations are induced by projective transformations and the odd permutations are induced by polarities.*

Proof: This is proved in [S1]. Here is a sketch. Suppose we apply the standard polarity to our flag triple. We then get the same vector representatives except that their roles have changed. Therefore, the triple product of the dual triple is the reciprocal of the original triple product. If we then apply an odd permutation to the flags we get back to the original invariant. This operation implies the existence of an order 2 symmetry of the flag, induced by a polarity, which does an odd permutation to the flats comprising the prism.

The 2-fold symmetry just explained combines with the 3-fold symmetry to give us a symmetry group $H = S_3$ of order 6. Suppose ψ is some other symmetry. Composing with some element of H we can consider the case when ψ preserves at least one flag and also is a projective transformation. But then ψ has to induce the identity permutation on the flags because of the triple invariant. But then ψ is a projective transformation which fixes 6 general position points. Hence ψ is the identity. This shows that H is the full group of symmetries. ♠

Lemma 4.3 *Let Π be a generic prism. Each order 2 isometry of Π fixes a unique point in the flat of Π that it stabilizes.*

Proof: This is proved in [S1]. Here is the proof again. Let δ be such an isometry and let F be the flat such that $\delta(F) = F$. The duality δ swaps the two flags defining F and hence reverses the directions of the medial geodesics foliating F and asymptotic to these flags. Also, being a polarity, δ reverses the directions of all singular geodesics in F . In particular δ reverses an orthogonal pair of directions. This forces δ to reverse every direction. If we identify F with \mathbf{R}^2 then δ is acting as an isometry whose linear part is an order 2 rotation. Such maps have unique fixed points in \mathbf{R}^2 . ♠

Definition: On a generic prism Π , the *inflection points* are the fixed points of the order 2 isometries of Π . There are 3 such inflection points, one per flat. They are permuted by the order 3 isometries of Π . The *inflection lines* are the singular geodesics which contain the inflection points and which are perpendicular to the geodesics in the medial foliations.

For a prism based on either of the two middle pictures in Figure 4.1, the inflection points do not exist. Geometrically, what is happening as we approach one of these prisms through a family of generic prisms is that the inflection points move off to ∞ . The inflection lines still exist however, as we now explain.

In [S1] we show that every Pappus modular group is an isometry group of an embedded pattern of flats. In the generic case we show that each fixed point of an order 2 element of the group is contained in the relevant inflection line. (See also 5.5.) If we exclude the totally symmetric Pappus modular groups, the remaining 1-parameter family of non-generic groups can be normalized so that they all involve the same flat. Taking a limit of the generic result, we can say that all the order 2 fixed points of all these groups in F lie on the same singular geodesic which is perpendicular to the medial foliation of F . This singular geodesic is the inflection line in F .

We have not yet discussed the totally symmetric case, the prism based on the lefthand picture in Figure 4.1. The associated prism has an infinite symmetry group. Referring to Figure 1, the projective transformation which extends the map $x \rightarrow rx$, for any $r \neq 0$, induces an isometry that preserves the prism. These isometries act nontrivially on the flats. In this case every associated triangle is isometric to a hyperbolic Farey triangle. The inflection lines are comprised of the symmetry points on each ideal hyperbolic triangle.

4.3 Triangle Foliations

Let Π be a prism. The order 3 isometries of Π preserve the medial geodesic foliations. Thus Π is foliated by *triangles*, triples of medial geodesics invariant under the order 3 isometries of Π . In the generic case, exactly one triangle of Π contains all 3 inflection points. In all cases, the triangles of Π are perpendicular to the inflection lines. In particular, this fact gives us a way to pick out canonical *inflection points* on each triangle, namely where the triangle intersects the inflection line. In the totally symmetric case, all the triangles are isometric to hyperbolic ideal triangles and hence isometric to each other. For the other prisms the situation is very different.

Lemma 4.4 *Let Π_1 and Π_2 be generic prisms and let τ_k be a triangle of Π_k for $k = 1, 2$. Then τ_1 and τ_2 are isometric to each other only if Π_1 and Π_2 are isomorphic prisms. If $\Pi_1 = \Pi_2 = \Pi$ then τ_1 and τ_2 are isometric if and only if they are permuted by the symmetry group of Π .*

Proof: An isometry taking τ_1 to τ_2 would have to map the flats of Π_1 to the flats of Π_2 . This proves the first statement. For the second statement, note that a projective symmetry of Π taking γ_1 to γ_2 must be in the symmetry group of Π . ♠

4.4 The Axis

In this section we prove a properness result about prisms that will come in handy when we analyze components of the representation variety. We first need to define what we mean the axis of a prism.

Lemma 4.5 *The fixed point set of the order 3 symmetries of a prism is a singular geodesic.*

Proof: This is a general fact about order 3 isometries. We can normalize so that the order 3 symmetries are as in Equation 5. The fixed point set is the singular geodesic corresponding to the standard ellipses of the form $E(a, a, a^{-2})$. ♠

We call the fixed point set of the order 3 symmetry the *axis* of the prism.

Our next result compares two geometric properties of prisms. The second of these quantities is related to the topology of the representation space \mathcal{R} . The two quantities compared here are like our two distance parameters s and d discussed in §2. Given a prism Π and a point $p \in \Pi$, we define $s(\Pi, p)$ be the distance from p to the inflection line in the flat of Π that contains p . At the same time, let $d(\Pi, p)$ be the distance from p to the axis of Π .

Lemma 4.6 (Properness) *Let $\{\Pi_n, p_n\}$ be any sequence of prisms. If $s(\Pi_n, p_n) \rightarrow \infty$ then also $d(\Pi_n, p_n) \rightarrow \infty$.*

Proof: We will suppose this false and derive a contradiction. That is, we suppose that $s(\Pi_n, p_n) \rightarrow \infty$ but $d(\Pi_n, p_n)$ stays bounded. We can normalize by isometries so that the point on the axis of Π_n closest to p_n is the origin of X . This means that the distance from p_n to the origin is uniformly bounded. But then the flat F_n of Π_n containing p_n intersects a uniformly bounded region of X . Since the order 3 isometries of Π_n fix the origin, we see that all flats of Π_n intersect a uniformly bounded region of X .

But then we can take a limit and get a prism $\Pi = \lim \Pi_n$. Since the inflection lines of Π exist and are unique, we see that the inflection lines of Π_n remain within a uniformly bounded region of X . But then we have a uniformly bounded distance from p_n to the relevant inflection line, a contradiction. ♠

4.5 Modular Group Representations

We say that a *prism pair* is a pair (Π, p) where Π is a prism and $p \in \Pi$. We mean here that p is a point in one of the flats comprising Π . We impose a cyclic order on Π , determined by the cyclic order on the flats. The order 3 symmetries of Π respect this order and the order 2 symmetries do not. One of the order 3 symmetries cycles the flats of Π one click forward in the cyclic order and the other one cycles the flats of Π one click backward. We prefer the former symmetry and we call it the *forward symmetry*. When we normalize as in Figure 4.1, the forward symmetry is given by the element $\rho(\sigma_3)$ from Equation 5

The prism pair (Π, p) determines a point in \mathcal{R} . We let $\rho(\Pi, p)$ be the representation such that $\rho(\sigma_2)$ is the elliptic polarity fixing p and $\rho(\sigma_3)$ is the forward symmetry of Π . We call these the *prism representations*.

We call two prism pairs (Π_1, p_1) and (Π_2, p_2) *equivalent* if there is an isometry of X which maps the first pair to the second and respects the imposed cyclic orders. We let \mathcal{A} denote the space of equivalence classes of prism pairs. We have a map $\mathcal{A} \rightarrow \mathcal{R}$. Here $\mathcal{R} \cong \mathbf{R}^3 - \{(0, 0, 0)\}$ is the big representation space we considered in the previous chapter. We call a prism pair (Π, p) *neutral* if p lies on an inflection line of Π . We proved in [S1] that every neutral prism pair gives rise to a Pappus representation of the modular group and conversely that every Pappus representation of the modular group arises this way. Also see §5.5.

We let $\mathcal{P} \subset \mathcal{A}$ denote the set of neutral prism pairs. Our results in the next chapter will show that the map $\rho : \mathcal{A} \rightarrow \mathcal{R}$ is one-to-one on \mathcal{P} and two-to-one on $\mathcal{A} - \mathcal{P}$. This result generalizes the folding phenomenon we discussed in §2 in the hyperbolic setting.

5 The Big Calculation

5.1 The Main Results

We continue the notation from the last section of the previous chapter. Given $\rho = \rho(\Pi, p)$ define

$$g_\rho = \rho(\sigma_2\sigma_3). \tag{8}$$

This element $g = g_\rho$ preserves one of the flags f_1 associated to the flat F_1 of Π that contains p . To see this, let f_1, f_2, f_3 be the flags defining Π , chosen so that F_1 is determined by the pair (f_1, f_2) . Then $\sigma_3(f_1) = f_2$ and $\sigma_2(f_2) = f_1$. Hence $g(f_1) = f_1$. The square g^2 also preserves f_1 . It is easier to work with g^2 because this element is a projective transformation. Below we prove the following result:

Theorem 5.1 *The element g_ρ^2 is parabolic iff p lies on the inflection line of Π . This happens iff ρ is a Pappus modular group representation. Otherwise g_ρ^2 has eigenvalues $(\lambda, 1/\lambda, 1)$ with $\lambda \in (-\infty, -1) \cup (-1, 0)$. We can choose λ so that the corresponding eigenvector corresponds to the flag f_1 .*

The final statement requires some explanation. To keep consistent with our notation below, we write $f_1 = (b_1, L_2)$. What we are saying, first of all, is that the eigenvector of g^2 corresponding to λ represents b_1 . We are also saying that λ is an eigenvalue of $(g^{-2})^t$, and the corresponding eigenvector represents L_2 .

We call the prism pair (Π_1, p_1) *attracting* if $|\lambda| > 1$. This property is independent of how we normalize (Π, p) . This is obvious if we replace (Π, p) by some pair $(T(\Pi), T(p))$ where T is a projective transformation. This is far less obvious if we take T to be a duality. The reader might worry that somehow λ gets changed to $1/\lambda$. This is not the case. One way to check this is just to try some experiments with diagonal matrices and the standard flags associated to them. Another way is to observe that the attracting nature of (Π, p) has a geometric interpretation in terms of the symmetric space X : The isometry g^2 is moving points in X towards the point in the visual boundary corresponding to f . This is an isometry-invariant way to talk about the attracting nature of (Π, p) . If $|\lambda| < 1$ we call (Π, p) *repelling*. Finally, as in the previous chapter, we call (Π, p) *neutral* if $\lambda = -1$.

The element g^2 also has an eigenvalue $1/\lambda$ and there is some other flag f'_1 that corresponds to this eigenvalue. below we also prove the following result.

Theorem 5.2 *If ρ is not a Pappus modular group representation, then the orbit of f'_1 under $\rho(\sigma_3)$ defines a prism Π' such that $\rho(\Pi', p) = \rho(\Pi, p)$. Exactly one of the prism pairs is attracting and exactly one is repelling.*

We will prove Theorems 5.1 and 5.2 in this chapter.

Recall that \mathcal{A} is the space of isometry classes of prism pairs. Let \mathcal{A}_+ denote the set of attracting prism pairs. Our corollary below favors the attracting prism pairs over the repelling prism pairs, but we could make the same kind of statement about the repelling pairs. Let $\rho : \mathcal{A} \rightarrow \mathcal{R}$ be the map which assigns each isometry class of prism pair its representation class in \mathcal{R} . We are slightly abusing notation here, because $\rho(\Pi, p)$ is also denoting the individual representation based on (Π, p) and not its conjugacy class.

Corollary 5.3 *The map ρ is injective on $\mathcal{P} \cup \mathcal{A}_+$ and $\rho(\mathcal{P} \cup \mathcal{A}_+) = \rho(\mathcal{A})$.*

Proof: Certainly a neutral prism pair cannot give the same representation as an attracting or repelling pair because parabolic elements are not conjugate to loxodromic elements. Hence $\rho(\mathcal{P}) \cap \rho(\mathcal{A}_+) = \emptyset$.

Suppose $\rho(P_1, p_1) = \rho(P_2, p_2)$ for two prism pairs in $\mathcal{P} \cup \mathcal{A}_+$. Such that $\rho(\Pi_1, p_1) = \rho(\Pi_2, p_2)$. (We can adjust by an isometry so that these representations are equal and not just conjugate.) The common element g^2 cannot be both loxodromic and parabolic. Hence both prism pairs are either neutral or attracting.

Consider the neutral case first. The element g^2 has a unique fixed flag f_1 , and f_1 must be one of the triple of flags defining both Π_1 and Π_2 . But then the orbit of f_1 under $\rho_k(\sigma_3)$ defines Π_k . Since $\rho_1(\sigma_3) = \rho_2(\sigma_3)$ we see that $\Pi_1 = \Pi_2$. Since $\rho_1(\sigma_2) = \rho_2(\sigma_2)$ and p_k is the unique fixed point of $\rho_k(\sigma_2)$, we have $p_1 = p_2$.

Now consider the attracting case. One of the flags f_1 defining Π_k is the attracting fixed point of g_k^2 for each $k = 1, 2$. Since these are the same element, the same flag is part of the triple defining both Π_1 and Π_2 . But then the orbit of this flag under the common element $\rho_k(\sigma_3)$ gives the triple defining Π_k . Hence $\Pi_1 = \Pi_2$. Likewise $p_1 = p_2$. This proves the first statement of the lemma.

The second statement follows from Theorem 5.2, which says that each non-neutral member of \mathcal{A} has the property that there are both attracting and repelling pairs which give the same representation. ♠

5.2 Normalizing Triples of Flags

As preparation for proving Theorems 5.1 and 5.2 we discuss how to normalize triples of flags. We consider the generic case, and then at the end of our calculations consider the non-generic case. We can normalize the picture as in the right-hand picture in Figure 4.1. Figure 5.1 repeats with this picture, and with labels. The flags are $f_k = (b_k, L_{k+1})$. Here and everywhere else we take the indices mod 3.

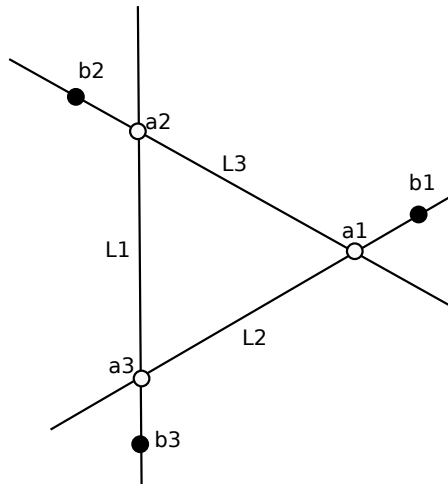


Figure 5.1: A normalized Flag

In the case shown in Figure 5.1, the point a_1 is between a_3 and b_1 on the line L_2 . This case corresponds to the triple invariant of (f_1, f_2, f_3) lying in $(-1, 0)$. The other case would be when a_3 lies between b_1 and a_1 . This corresponds to the triple invariant lying in $(-\infty, -1)$. The intermediate case, when b_1, b_2, b_3 all lie on the line at infinity, corresponds to the triple invariant being equal to -1 .

We can apply the standard duality Δ to the picture. The new flags $\Delta(f_1), \Delta(f_2), \Delta(f_3)$ have two properties we remark on:

1. The order 3 counterclockwise rotation about the origin has the action $\Delta(f_k) \rightarrow \Delta(f_{k+1})$.
2. The triple invariant of these new flags is the reciprocal of the triple invariant of the original flags.

What this means is that if we have a generic prism, we can always normalize it so that the corresponding flags are as in Figure 5.1, with a_1 between a_3 and b_1 . See §5.6 for more details.

5.3 The Big Calculation

In this section will compute g^2 , the element from Equation 8, and deduce information from the computation. We first treat the generic case, and then discuss the non-generic cases at the end of the section. We normalize as in Figure 5.1.

The Flags: We represent our points by 3-vectors in Mathematica:

$$a_1 = (1, 0, 1) \quad a_2 = (-1/2, \sqrt{3}/2, 1), \quad a_3 = (-1/2, -\sqrt{3}/2, 1). \quad (9)$$

The lines in Figure 5.1 are represented by the cross products $L_k = a_{k-1} \times a_{k+1}$. Next, we choose $t > 0$ and define

$$b_k = (1+t)a_k - ta_{k-1}. \quad (10)$$

Our flags are (b_k, L_{k+1}) for $k = 1, 2, 3$. The flag fixed by g^2 will, as above, be $f_1 = (b_1, L_2)$. The triple invariant of f_1, f_2, f_3 is

$$-\frac{t^3}{(t+1)^3} \in (-1, 0). \quad (11)$$

The Order 3 Element: The element $\rho(\sigma_3)$ is represented by the matrix

$$M_3 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (12)$$

This map has order 3 and has the action $f_k \rightarrow f_{k+1}$.

The Order 2 Element: Δ be the standard polarity. Let S be the matrix whose column vectors are $2rb_1, 2sb_2, a_1$:

$$S = \begin{pmatrix} r(2+3t) & -s(1+3t) & 1 \\ \sqrt{3}rt & \sqrt{3}s(1+t) & 0 \\ 2r & 2s & 1 \end{pmatrix} \quad (13)$$

Let L_X and L_Y respectively denote the lines in \mathbf{P} extending the X -axis and the Y -axis. Let $p_X \in L_X$ and $p_Y \in L_Y$ be the points at infinity. The duality Δ interchanges the flags (p_X, L_X) and (p_Y, L_Y) and the projective transformation represented by S carries these flags to f_1 and f_2 . The composition

$$\rho(\sigma_2) = S \circ \Delta \circ S^{-1} = \Delta \circ M_2 \quad M_2 = (S^{-1})^t S^{-1}. \quad (14)$$

gives the general form of the elliptic polarity which interchanges f_1 and f_2 . Thus, choosing the parameters (r, s) picks out a generating point in the flat F determined by these two flags. We compute

$$\det(S) = 6\sqrt{3}rst(1+t). \quad (15)$$

Since $t > 0$, this determinant is nonzero as long as $rs \neq 0$.

Now we observe a symmetry. If D is any diagonal matrix whose diagonal entries belong to the 2 element set $\{-1, +1\}$ then $D \circ \Delta = \Delta \circ D$. For this reason, the matrix $S \circ D$ gives the same polarity as the matrix S . This means that all the possibilities are covered by the cases $r, s > 0$.

The Key Element: Finally, we have

$$g^2 = (M_2^{-1})^t (M_3^{-1})^t M_2 M_3. \quad (16)$$

To see why this works, we work from right to left. We start out with a vector representing a point. We apply M_3 and we get another vector representing a

point. Now we apply ΔM_2 and we get a vector representing a line. The next two matrix operations involve the inverse transpose because we are acting on lines. Finally, we apply Δ and we get a vector representing a point.

As a sanity check, we compute that

$$\det(g^2) = 1, \quad g^2(b_1) = b_1, \quad g^2(L_2) = L_2. \quad (17)$$

For the final calculation, of course, we use the inverse transpose of the matrix representing g^2 . Thus g^2 fixes the flag $f_1 = (b_1, L_2)$, as expected.

Eigenvalues: Now a miracle occurs. The matrix g^2 is huge, but we compute in Mathematica that its eigenvalues are:

$$1, \quad \lambda, \quad \lambda^{-1}, \quad \lambda = -\left(\frac{r^2}{s^2}\right)\left(\frac{t}{1+t}\right) < 0. \quad (18)$$

This element is loxodromic unless $\lambda = -1$. The eigenvector corresponding to λ represents b_1 .

Exploring the Dichotomy: We have $r, s, t > 0$. From the calculation above, we see that g^2 is loxodromic unless

$$r = \mu s, \quad \mu = \sqrt{\frac{1+t}{t}}. \quad (19)$$

We hold t fixed for the rest of this subsection. With t fixed, the parabolic case is parametrized by the infinite set $s > 0$, which is homeomorphic to a line. Call this set \mathcal{J} .

Now let us look at the Pappus representations. In [S1] we prove that these representations correspond to prism pairs (Π, p) with p on an inflection line. We give a different proof below in §5.5. The triple invariant of the representation is an injective function of our parameter t . With t fixed, we get an iso-prismatic family parametrized by the inflection line in F : The prisms are all the same. Call this family \mathcal{I} . Each member of \mathcal{I} gives us a triple (r, s, t) , and this triple must lie in \mathcal{J} because the corresponding g^2 is parabolic. This gives us a map $\mathcal{I} \rightarrow \mathcal{J}$. No two distinct representations in \mathcal{I} are conjugate to each other, because a conjugacy would preserve the pattern of flats, prisms, triangles, and inflection points. Hence, the map $\mathcal{I} \rightarrow \mathcal{J}$ is injective. The family \mathcal{I} is parametrized by the point p on the inflection line. As p exits every compact subset of X , the corresponding parabolic element

exits every compact subset of $SL_3(\mathbf{R})$. Hence the map $\mathcal{I} \rightarrow \mathcal{J}$ is proper. Hence $\mathcal{J} = \mathcal{I}$. This proves Theorem 5.1 in the generic case.

The Second Prism Description Consider the flag $f'_1 = (b'_1, L'_2)$ corresponding to the eigenvalue $1/\lambda$ of g^2 and $(g^{-2})^t$. The flag f'_1 is distinct from f_1 because the eigenvalues are different. The eigenflag f'_1 has a monstrous formula, but the coordinates are rational functions of r, s, t . We get a new triple of flags by taking the orbit of (p'_1, L'_2) under the action of M_3 . Thus $p'_{k+1} = M_3(p'_k)$ and $L_{k+1} = M_3(L'_k)$. Normally we would use the inverse-transpose to compute the new lines, but in this case $(M_3^{-1})^t = M_3$.

The new triple of flags in turn defines a new prism Π' together with a new flat F' of Π' corresponding the flags (b'_1, L'_2) and (b'_2, L'_3) . We compute (in Mathematica) that $\rho(\sigma_2)$ swaps (b'_1, L'_2) with (b'_2, L'_3) . This means that the fixed point of $\rho(\sigma_2)$ in X , namely the generating point p for our representation, also lies in F' . So, the pair (Π', p) is a second description of the same prism group. Exactly one prism pair is attracting and one is repelling. This proves Theorem 5.2 in the generic case.

The Non-Generic Cases: For the totally symmetric case we are back in the hyperbolic plane with the Farey triangulation and its shears. In this case, Theorem 5.1 follows from the hyperbolic geometry picture developed in §2.

The remaining cases correspond to the case when the triple invariant is -1 but the triple is not completely symmetric. In this case we set $b_k = a_k$ for $k = 1, 2, 3$, and L_1, L_2, L_3 are the line through the origin with $b_k \in L_{k+1}$. The matrix S above is now the one whose column vectors are $2rb_1, 2rb_2, (0, 0, 1)$. With these changes, the calculation above, and all the results, go through just as in the generic case.

5.4 Comparing the Prisms

We consider the loxodromic case in more detail. We call the two prism pairs (Π, p) and (Π', p) *partners*. Equation 11 gives the triple invariant for the flags defining Π . The invariant for Π is $3 \log((t+1)/t)$. Let τ' be the triple invariant for the flags defining Π' . See §5.6.1 below for the monstrous formula. Inspecting this formula, we see that $\tau' < 0$ no matter which $r, s, t > 0$ we choose. See the discussion at the end of §5.6.1. As in Equation 11, the expression for τ' is a perfect cube.

Here is a sample calculation. The prism invariants for Π and Π' when $(r, s, t) = (1, 1, 1)$ are respectively

$$3 \log(2), \quad 3 \log\left(\frac{9825}{5602}\right)$$

In the non-generic case, the formula for τ' is shorter:

$$-\frac{(36r^{12}s^2 + r^{12} + 3r^{10}s^2 + 3r^8s^4 + 1296r^6s^{10} + 144r^6s^8 + 2r^6s^6 + 108r^4s^{10} + 3r^4s^8 + 3r^2s^{10} + s^{12})^3}{(r^{12} + 1296r^{10}s^6 + 108r^{10}s^4 + 3r^{10}s^2 + 144r^8s^6 + 3r^8s^4 + 2r^6s^6 + 3r^4s^8 + 36r^2s^{12} + 3r^2s^{10} + s^{12})^3}$$

One can see, again, that this expression is always negative. When $r = s$ the expression equals -1 as it must in the parabolic case. In general, the expression can take on all negative values. Reversing the roles played by Π and Π' we see that a non-generic pair (Π', p) can arise from a generic pair (Π, p) no matter what the prism invariant of Π . It all depends on the choice of p .

5.5 Pappus Representations and the Inflection Line

In this section we prove, as we did in [S1], that the Pappus representations correspond to prism pairs (Π, p) where p lies on the inflection line of the flat F which contains p . Our proof is a bit different here. In §6.2 we prove:

Lemma 5.4 *For each prism Π there exists a Pappus modular group whose order 2 fixed points are the inflection points of Π .*

Recall that the flat F is foliated by medial geodesics defined by the flags (b_1, L_2) and (b_2, L_3) . The orthogonal foliation is by singular geodesics which we call *orthogonal singular geodesics*.

Lemma 5.5 *The following is true.*

1. *The points (r, s, t) and (rd, sd, t) parametrize representations which are $\sqrt{2/3}\log(d)$ units apart along an orthogonal singular geodesic.*
2. *The points (r, s, t) and $(rd, s/d, t)$ parametrize representations which are $\sqrt{2}\log(d)$ units apart along a medial geodesic.*

Proof: Referring to Equation 14, let $I_0 = M_2(r, s, t)$ and $I_1 = M_2(rd, sd, t)$. Let

$$J = (I_1^{-1})^t I_0.$$

Then J is a translation along F along the vector which is twice the difference between the fixed points of I_0 and I_1 . We compute that the eigenvalues of J are $(1, d^2, d^2)$, and the vectors representing b_1 and b_2 are both eigenvectors corresponding to d^2 . This shows that J is translation along the singular geodesics by $\sqrt{8/3} \log d$ units. (We get the funny factor because we compute the distance by first rescaling J so that it belongs to $SL_3(\mathbf{R})$.) Hence the fixed points of I_0 and I_1 are $\sqrt{2/3} \log(d)$ units apart and on the same orthogonal singular geodesic. This proves Statement 1.

For Statement 2, we set $I_1 = M_2(rd, s/d, t)$. We compute that the eigenvalues of $J \in SL_3(\mathbf{R})$ are $(d^2, d^{-2}, 1)$, and the vectors representing b_1 and b_2 are eigenvectors respectively corresponding to d^2 and $1/d^2$. Finally, the point $a_1 = L_2 \cap L_3$ is the eigenvector corresponding to 1. Our result follows from this structure. ♠

Our claim about the inflection line follows from Equation 19, Lemma 5.4, and Statement 1 of Lemma 5.5. Statement 2 of Lemma 5.5 is not needed until later.

5.6 Symmetries of the Parametrization

Now we discuss some of the symmetries of our (r, s, t) parametrization.

First Symmetry: Define

$$\iota_1(r, s, t) = \left(\frac{r_0^2}{r}, \frac{s_0^2}{s}, t \right). \quad (20)$$

Here r_0 and s_0 depend on t but we have suppressed the dependence. Note that the map ι_1 is an involution which fixes (r_0, s_0, t) . If we use log-coordinates and identify the (r, s) positive quadrant with \mathbf{R}^2 , then this map is the order 2 isometry of \mathbf{R}^2 which fixes the point corresponding to the axial representation. This map corresponds to the action of the order 2 symmetry of the prism Π which preserves the flat $F \cong \mathbf{R}^2$ containing the generating point. The representations corresponding to (r, s, t) and $\iota_1(r, s, t)$ are not conjugate,

but they have the same image in $\text{Isom}(X)$. The representations become conjugate if we switch the order 3 generator of one of the groups to its inverse.

Second Symmetry: In §5.2 we said that we would restrict our attention to triples of flags with triple invariant in $(-1, 0)$. This corresponds to $t \in (0, \infty)$. The case of flags with triple invariant in $(-\infty, -1)$ corresponds to $t \in (-\infty, -1)$. We mentioned that we can always arrange the former case by applying a duality to our initial triple of flags. We define

$$\iota_2(r, s, t) = \left(\frac{s_0^2}{s}, \frac{r_0^2}{r}, -1 - t \right). \quad (21)$$

The two sets of parameters (r, s, t) and $\iota_2(r, s, t)$ describe groups that are conjugate *via* a duality. We call ι_2 the *duality involution*.

5.6.1 The Monster Expression

The expression for the second triple invariant τ' is $-(1+t)^3 A^3 / (t^3 B^3)$, where A and B respectively are:

$$\begin{aligned} & 576r^{12}s^2t^8 + 1152r^{12}s^2t^7 + 960r^{12}s^2t^6 + 384r^{12}s^2t^5 + 64r^{12}s^2t^4 + 16r^{12}t^8 + \\ & 48r^{10}s^2t^8 + 192r^{10}s^2t^7 + 288r^{10}s^2t^6 + 192r^{10}s^2t^5 + 48r^{10}s^2t^4 + 48r^8s^4t^8 + \\ & 96r^8s^4t^7 + 48r^8s^4t^6 + 12r^8s^2t^6 + 24r^8s^2t^5 + 12r^8s^2t^4 + 20736r^6s^{10}t^8 + \\ & 82944r^6s^{10}t^7 + 152064r^6s^{10}t^6 + 165888r^6s^{10}t^5 + 117504r^6s^{10}t^4 + 55296r^6s^{10}t^3 + \\ & 16896r^6s^{10}t^2 + 3072r^6s^{10}t + 256r^6s^{10} + 2304r^6s^8t^8 + 4608r^6s^8t^7 + 3840r^6s^8t^6 + \\ & 1536r^6s^8t^5 + 256r^6s^8t^4 + 32r^6s^6t^8 - \mathbf{192r^6s^6t^7} - \mathbf{192r^6s^6t^6} - \mathbf{64r^6s^6t^5} + 16r^6s^4t^6 + \\ & r^6s^2t^4 + 1728r^4s^{10}t^8 + 10368r^4s^{10}t^7 + 27072r^4s^{10}t^6 + 40320r^4s^{10}t^5 + 37632r^4s^{10}t^4 + \\ & 22656r^4s^{10}t^3 + 8640r^4s^{10}t^2 + 1920r^4s^{10}t + 192r^4s^{10} + 48r^4s^8t^8 + 192r^4s^8t^7 + 288r^4s^8t^6 + \\ & 192r^4s^8t^5 + 48r^4s^8t^4 + 48r^2s^{10}t^8 + 384r^2s^{10}t^7 + 1344r^2s^{10}t^6 + 2688r^2s^{10}t^5 + \\ & 3360r^2s^{10}t^4 + 2688r^2s^{10}t^3 + 1344r^2s^{10}t^2 + 384r^2s^{10}t + 48r^2s^{10} + 16s^{12}t^8 + \\ & 96s^{12}t^7 + 240s^{12}t^6 + 320s^{12}t^5 + 240s^{12}t^4 + 96s^{12}t^3 + 16s^{12}t^2 + 4s^{10}t^6 + \\ & 24s^{10}t^5 + 60s^{10}t^4 + 80s^{10}t^3 + 60s^{10}t^2 + 24s^{10}t + 4s^{10} \end{aligned}$$

$$\begin{aligned}
& r^2s^6 + 16r^4s^6 + 96r^6s^6 + 256r^8s^6 + 256r^{10}s^6 + 16s^{12} + 64r^2s^{12} + \\
& 4r^2s^6t + 96r^4s^6t + 768r^6s^6t + 2560r^8s^6t + 3072r^{10}s^6t + 128s^{12}t + \\
& 640r^2s^{12}t + 6r^2s^6t^2 + 240r^4s^6t^2 + 2688r^6s^6t^2 + 11520r^8s^6t^2 + 16896r^{10}s^6t^2 + \\
& 12r^2s^8t^2 + 48r^4s^8t^2 + 448s^{12}t^2 + 2880r^2s^{12}t^2 + 4r^2s^6t^3 + 320r^4s^6t^3 + 5312r^6s^6t^3 + \\
& 30208r^8s^6t^3 + 55296r^{10}s^6t^3 + 48r^2s^8t^3 + 288r^4s^8t^3 + 896s^{12}t^3 + 7552r^2s^{12}t^3 + \\
& 48r^8s^4t^4 + 192r^{10}s^4t^4 + r^2s^6t^4 + 240r^4s^6t^4 + 6400r^6s^6t^4 + 50176r^8s^6t^4 + \\
& 117504r^{10}s^6t^4 + 72r^2s^8t^4 + 720r^4s^8t^4 + 48r^2s^{10}t^4 + 1120s^{12}t^4 + 12544r^2s^{12}t^4 + \\
& 192r^8s^4t^5 + 1152r^{10}s^4t^5 + 96r^4s^6t^5 + 4736r^6s^6t^5 + 53760r^8s^6t^5 + 165888r^{10}s^6t^5 + \\
& 48r^2s^8t^5 + 960r^4s^8t^5 + 192r^2s^{10}t^5 + 896s^{12}t^5 + 13440r^2s^{12}t^5 + 4r^{10}t^6 + \\
& 16r^{12}t^6 + 288r^8s^4t^6 + 2880r^{10}s^4t^6 + 16r^4s^6t^6 + 2048r^6s^6t^6 + 36096r^8s^6t^6 + \\
& 152064r^{10}s^6t^6 + 12r^2s^8t^6 + 720r^4s^8t^6 + 288r^2s^{10}t^6 + 448s^{12}t^6 + 9024r^2s^{12}t^6 + \\
& 32r^{12}t^7 + 192r^8s^4t^7 + 3456r^{10}s^4t^7 + 448r^6s^6t^7 + 13824r^8s^6t^7 + 82944r^{10}s^6t^7 + \\
& 288r^4s^8t^7 + 192r^2s^{10}t^7 + 128s^{12}t^7 + 3456r^2s^{12}t^7 + 16r^{12}t^8 + 48r^{10}s^2t^8 + \\
& 48r^8s^4t^8 + 1728r^{10}s^4t^8 + 32r^6s^6t^8 + 2304r^8s^6t^8 + 20736r^{10}s^6t^8 + 48r^4s^8t^8 + \\
& 48r^2s^{10}t^8 + 16s^{12}t^8 + 576r^2s^{12}t^8
\end{aligned}$$

Remarks:

- (1) I used chatGPT 4o mini to help me format these monsters.
- (2) We have $B > 0$ for all r, s, t because B is the sum of positive monomials. When $r \geq 1$ and $t \geq 1$ and $s \leq 1$, the first term of A dominates the sum of the bolded negative terms, and so $A > 0$ in this case. There are 7 other cases, depending on the signs of $r - 1, s - 1, t - 1$, and in all cases one can find short sums of positive terms which dominate the 3 bolded terms. Hence $A > 0$ for all $r, s, t > 0$. Hence $\tau' < 0$ for all $r, s, t > 0$.

6 Recognizing the Representations

6.1 The Pappus Modular Groups

In this chapter we relate the prism representations to the Pappus modular groups and to the Anosov representations considered in [BLV] and [S2]. This section is a near-repeat of the corresponding section in [S2]. One subtle difference is that we choose the parameters c, d in $(-1, 1)$ as in [S2] rather than in $(0, 1)$ as in [S1].

Convex Marked Boxes: A *convex marked box* is a convex quadrilateral in \mathbf{P} together with a distinguished point in the interior of one side and a distinguished point in the interior of an opposite side. We call one of the points the *top* point and the other one the *bottom* point. Correspondingly

we call the edges containing these points the *top edge* and the *bottom edge*. Finally, we say that the *top flag* is the flag (p, ℓ) where p is the top point and ℓ is the line extending the top edge. We define the *bottom flag* similarity.

Operations on Marked Boxes: There are 3 operations we can perform on marked boxes, and we call them t, b, i . Figure 6.1 shows how they act.

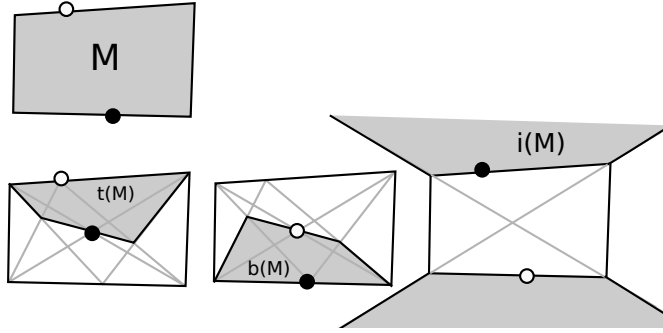


Figure 6.1: The three operations on marked boxes packing

These operations satisfy the relations

$$i^2 = I, \quad tit = b, \quad bib = t, \quad tibi = I, \quad biti = I. \quad (22)$$

here I is the identity. As a consequence of these relations, and the nesting of the marked boxes. The group of operations isomorphic to the modular group. The explicit generators are (say) i and ti . We let \mathcal{M} be the orbit of a marked box under the action of this group.

Order Three Symmetries of the Orbit: Given a marked box $M \in \mathcal{M}$ there is an order 3 projective transformation T_M which has the orbit

$$i(M) \rightarrow t(M) \rightarrow b(M).$$

This accounts for the order 3 elements of the Pappus modular groups. If we list out the top and bottom flats of these three marked boxes, they coincide in pairs and we end up with a triple of flags. The triple always turns out to be negative. Thus each marked box M in \mathcal{M} gives us a prism in X .

Order Two Symmetries of the Orbit: There is also an elliptic polarity which, in a certain sense, swaps M and $i(M)$. To make sense of this, we have to recall the notion of a *doppelganger* defined in [S1].

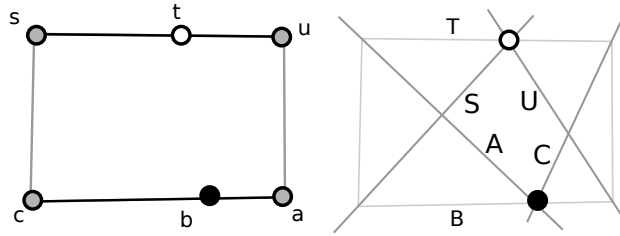


Figure 6.2 A convex marked box and its doppelganger

The 6-tuple (s, t, u, a, b, c) shown on the left side of Figure 6.2 encodes the marked box M . Here t and b are respectively the top and bottom points of M . The corresponding 6-tuple of lines (S, T, U, A, B, C) , which is defined entirely in terms of M , encodes a convex marked box M^* in \mathbf{P}^* . We can repeat the operation and we get $M^{**} = M$. It turns out that the i, b, t operations commute with the doppelganger operation and we can think of our orbit \mathcal{M} as an orbit of pairs of the form (M, M^*) . We call such a pair an *enhanced convex marked box*.

We showed in [S1] that there is an elliptic polarity δ_M that swaps M and $(i(M))^*$, and simultaneously swaps M^* and $i(M)$.

Putting it Together: Given a marked box m we have 3 flags coming from the tops and bottoms of $i(m)$, $t(m)$ and $b(m)$. We let Π be the corresponding prism. To get a prism representation ρ , we let $\rho(\sigma_2) = \delta_M$ and we let $\rho(\sigma_3)$ be the order 3 symmetry discussed above. By construction $\rho(\sigma_3)$ permutes the flats of Π , and the fixed point p of $\rho(\sigma_2)$ lies in flat F of Π determined by the tops and bottoms of $i(m)$.

6.2 The Axial Case

We define an equivalence relation on real numbers, $(c, d) \sim (-c, -d)$. We denote the equivalence class of (c, d) by $[(c, d)]$. A marked box M has an invariant $[M] = [(c, d)]$ where c and d are defined as follows. We normalize by a projective transformation so that the quadrilateral underlying M has vertices $(\pm 1, \pm 1)$. The top point then has coordinates $(c, 1)$ and the bottom point has coordinates $(d, -1)$. The reason for the equivalence relation is that we might have normalized by post-composing our projective transformation by the map which is reflection in the vertical line $x = 0$.

We say that M is *axial* if $[M] = [(0, d)]$ or $[M] = [(c, 0)]$. Here is a sharper version of Lemma 5.4. We also prove this result in [S1].

Lemma 6.1 *Let M be an axial marked box and let (Π, p) be the prism pair corresponding to the associated Pappus modular group representation. Then p is an inflection point of Π .*

Proof: We treat the case when $[M] = [(0, d)]$. The other case follows from duality: $[i(M)] = [(d, 0)]$.

Two marked boxes M_1 and M_2 are projectively equivalent if and only if $[M_1] = [M_2]$. Also, if $[M] = [(c, d)]$ then $i(M) = [(-d, c)]$. This is a direct calculation² which we do in [S1]. So, if $[M] = [(0, d)]$ then, as we already mentioned, $i(M) = [(-d, 0)] = [(d, 0)]$. Hence there is a projective transformation ϕ which maps M to $i(M)$, switching the tops and bottoms. Note that ϕ^2 fixes all the vertices of M , and hence ϕ^2 is the identity.

For notational convenience set $h = \rho(\sigma_2)$. Define M' to be the marked box we get by simply exchanging what we mean by the top and the bottom. We let $\psi = h \circ \phi$. Then ψ is a duality which maps M to the doppelganger of M' . So, ψ maps $t(M)$ and $b(M)$ respectively to the doppelgangers of $b(M)$ and $t(M)$. In particular, ψ permutes the 3 flags associated to $i(M), t(M), b(M)$. Hence $\psi(\Pi) = \Pi$. We have identified ψ as symmetry of Π . Since ψ is a duality, ψ has order 2 and fixes one of the inflection points of Π .

Since $h = h^{-1}$ we have $\phi = h\psi$. But then $h\psi h\psi$ is the identity. Hence $h\psi = \psi h$. Hence ψ and h have the same fixed point. Since the fixed point of ψ is an inflection point of Π , the fixed point of h is the same inflection point. ♠

6.3 The Space of Prism Representations

Two Pappus modular group representations are conjugate if and only if the enhanced marked boxes in their orbits are projectively equivalent, either by dualities or projective transformations. We can get a section of the space of Pappus representations by normalizing so that our initial marked box is the square Q with vertices ± 1 and the top point t lies in the interior of the top edge of Q and the bottom point b lies in the interior of the bottom edge of Q .

We let $M(c, d)$ be the marked box which is normalized this way and has $[M] = [(c, d)]$. The boxes $M(c, d)$ and $M(c', d')$ define the same representation in \mathcal{R} if and only if (c, d) and (c', d') are in the same ρ -orbit, where ρ is

²Again we mention that we normalized so that $(c, d) \in (0, 1)$ in [S1]. Here we are taking $(c, d) \in (-1, 1)$ and so the formulas are a bit different.

the order 4 rotation about the origin. This, as we saw in [S1], the space of Pappus modular group representations is homeomorphic to the cone

$$\mathcal{C} = (-1, 1)^2 / \langle \rho \rangle. \quad (23)$$

This space is in turn homeomorphic to \mathbf{R}^2 .

Let h be the map which assigns to each point in \mathcal{C} the isometry class of prism pairs for the associated Pappus modular group representation. The map h is a homeomorphism between \mathcal{C} and \mathcal{P} .

Lemma 6.2 *The map h extends to be a homeomorphism from $\mathcal{C} \times [0, \infty)$ to $\mathcal{P} \cup \mathcal{A}_+$.*

Proof: Given $c \in \mathcal{C}$ and $r \in [0, \infty)$ let (Π, p_0) be the prism pair given by $h(c)$. Let F be the flat of Π containing p_0 . Recall that p_0 lies on the inflection line in F . Let γ be the geodesic in the medial geodesic foliation of F that contains p_0 . One component of $\gamma - p_0$ consists of points q such that (Π, q) is attracting and the other component is the repelling case. These components cannot mix because the only neutral pairs lie on the inflection line. So, we let q_r be the unique point of γ such that $d_X(p_0, p_r) = r$ and (Π, p_r) is an attracting pair.

By construction, h gives a continuous proper bijection from $\mathcal{C} \times [0, \infty)$ to $\mathcal{P} \cup \mathcal{A}_+$. A map with all these properties is a homeomorphism. The continuity follows from the fact that if (Π, p_0) and (Π', p'_0) are two nearby prism pairs, then the points p_r and p'_r are also close in X . The key observation is that the attracting rays of γ and γ' point in the about the same rather than about opposite directions. The properness follows from the fact that, as $r \rightarrow \infty$, the distance from p_r to the inflection line of Π tends to ∞ . ♠

6.4 The Image in the Big Representation Space

Let $O = (0, 0, 0)$, the origin in \mathbf{R}^3 . We have a composition of maps

$$\mathbf{R}^2 \times [0, \infty) \rightarrow \mathcal{P} \cup \mathcal{A}_+ \rightarrow \mathcal{R} \cong \mathbf{R}^3 - O. \quad (24)$$

The first of these maps is a homeomorphism. The second of these maps is both continuous and injective. Therefore, the composition

$$f : \mathbf{R}^2 \times [0, \infty) \rightarrow \mathbf{R}^3 - O \quad (25)$$

is continuous and injective.

Since f is a map between 3-manifolds, it follows from Invariance of Domain that $f(\mathbf{R}^2 \times (0, \infty))$ is an open subset of $\mathbf{R}^3 - O$ and f is a homeomorphism from this set onto its image. We also remark that the image of f stays outside a neighborhood of O because representations indexed by points close to O satisfy $\text{trace}(g^2) \sim 0$, and all prism representations have $\text{trace}(g^2) \leq -1$. Here g^2 is the element we have studied extensively in the previous chapter.

Lemma 6.3 *f is a proper map.*

Proof: What we mean is that if $\{q_n\}$ is a sequence of points in $\mathbf{R}^2 \times [0, \infty)$ that exits every compact subset, then $\{f(q_n)\}$ also exits every compact subset of $\mathbf{R}^3 - O$. Since our image avoids a neighborhood of O we are really saying the the image sequence exits every compact subset of \mathbf{R}^3 . We suppose not and derive a contradiction.

Let (Π_n, p_n) be the prism pair associated to q_n . Let $\eta(\Pi_n, p_n)$ be the invariant computed in §4.4. The Properness Theorem tells us that if we have $\eta(\Pi_n, p_n) \rightarrow \infty$ then we also have $\nu(\Pi_n, p_n) \rightarrow \infty$. But this latter quantity is the distance in X from p_n , the fixed point of the element $\rho_n(\sigma_2)$, to the geodesic fixed by the element $\rho_n(\sigma_3)$. Here we are setting $\rho_n = \rho(\Pi_n, p_n)$. If this distance tends to ∞ then our representations exit every compact subset of \mathcal{R} . We conclude that $\{\eta(\Pi_n, p_n)\}$ remains uniformly bounded.

We want to see that in this case we also have $\nu(\Pi_n, p_n) \rightarrow \infty$. Since $\{q_n\}$ is exiting every compact subset of $\mathbf{R}^2 \times [0, \infty)$ it means that the first two coordinates of q_n are exiting every compact subset of \mathbf{R}^2 . The corresponding Pappus modular groups are exiting every compact subset of \mathcal{P} . Since there is a uniform bound between p_n and the point on the inflection line contained in the same medial geodesic, it suffices to prove our result when $\rho(\Pi_n, p_n)$ is a Pappus representation. We proved this result already in [S2, §4]. ♠

The image of \mathcal{A} in \mathcal{B} lies on the side of \mathcal{P} which has no elliptic elements. Since our map f is proper and injective, the image of \mathcal{A} in \mathcal{R} is precisely \mathcal{B} .

7 Patterns of Geodesics and Shearing

In this chapter we prove Theorems 1.2 and 1.3.

7.1 The Main Argument

Given the result at the end of the last chapter, we have 3 different descriptions of the Barbot component \mathcal{B} .

0. \mathcal{B} is the component of representations produced by the box morphing construction in [BLV] when it is extended as in [S2]. From this description, every element of \mathcal{B} is either Pappus or Anosov.
1. $\mathcal{B} = \mathcal{A}_+ \cup \mathcal{P}$. We can make this identification because the map from $\mathcal{A} \cup \mathcal{P}$ is a homeomorphism. In particular, every element of \mathcal{B} is a prism group.
2. $\mathcal{B} = \mathcal{A}_- \cup \mathcal{P}$. We can make this identification because the map from $\mathcal{A} \cup \mathcal{P}$ is a homeomorphism. In particular, every element of \mathcal{B} is a prism group in a different way.

The space $\mathcal{P} \cup \mathcal{A}_+$ has a foliation by rays, coming from the medial geodesics which naturally foliate the prisms. Identification 1 transfers this foliation to a foliation of \mathcal{B} by rays. The space $\mathcal{P} \cup \mathcal{A}_-$ also has a foliation by rays. Identification 2 transfers this foliation to a second foliation of \mathcal{B} by rays. Our analysis of the triple invariants shows that generically these two foliations of \mathcal{B} do not coincide.

Let $\rho \in \mathcal{B}$. When we use Identification 1, we produce prism description (Π_1, p_1) of ρ . When we use Identification 2, we produce a second prism description (Π_2, p_2) of ρ . Our analysis of the triple invariants shows that generically Π_1 and Π_2 are not isometric to each other. When $\rho \in \mathcal{P}$, the two descriptions coincide.

The prism Π_k contains a distinguished triangle γ_k , namely the one that contains the generating point p_k . The orbit of γ_k under Γ is the pattern $Y_{\rho,k}$. Here we have set $\Gamma = \rho(\mathbf{Z}/2 * \mathbf{Z}/3)$, as in Theorems 1.2 and 1.3. By construction,

$$\Gamma \subset \text{Isom}(Y_{\rho,1}), \quad \Gamma \subset \text{Isom}(Y_{\rho,1}).$$

The group Γ acts transitively on the prisms associated to $Y_{\rho,k}$. Hence each coset of Γ in $\text{Isom}(Y_{\rho,k})$ has a representative which is a symmetry of Π_k . This shows that our containment of groups has index at most 6. Generically, the symmetries of Π_k do not preserve the set of inflection points of Π_k . For this reason, $\Gamma = \text{Isom}(Y_{\rho,k})$ in the generic case.

Since Π_1 and Π_2 are not generically isometric, neither are the patterns $Y_{\rho,1}$ and $Y_{\rho,2}$. We also have the patterns $F_{\rho,1}$ and $F_{\rho,2}$ which are the Farey patterns

corresponding to the endpoints of the two rays containing our representation. The same triple invariant analysis shows that $F_{\rho,1}$ and $F_{\rho,2}$ are not isometric to each other. We have not yet pinned down the relationship between $F_{\rho,k}$ and $Y_{\rho,k}$. We will show below that the relationship is that of shearing.

To finish the proof we need to show that the patterns $Y_{\rho,k}$ are embedded for $k = 1, 2$, and we need to explain the shearing relationship between $F_{\rho,k}$ and $Y_{\rho,k}$. Finally, we need to establish the equality of the shearing strengths. The rest of the chapter is devoted to these things.

7.2 Pairs of Flags

In this section we give some preliminary information about certain pairs of flags.

We say that a pair of flags is *orthogonal* if the flat it determines contains the origin. In this case, the standard polarity switches the two flags. Figure 7.1 shows a typical orthogonal pair, (p_1, L_1) and (p_2, L_2) . The lines L_1 and L_2 are parallel, the line $\overline{p_1 p_2}$ contains the origin and is perpendicular to L_1 and L_2 . Finally, we have $d_1 d_2 = 1$ where d_k is the distance from L_k to the origin.

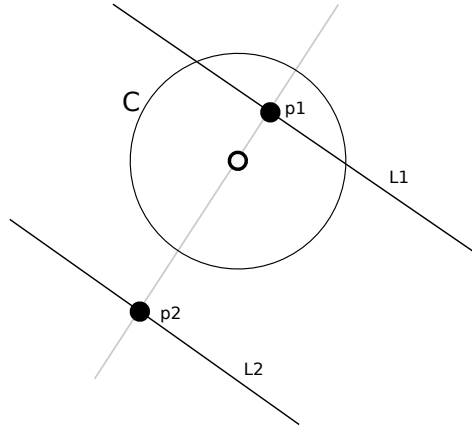


Figure 7.1 An orthogonal pair of flags and the unit circle C .

If we have two orthogonal pairs a_1, a_2 and b_1, b_2 then there are 8 triple invariants we can compute. In all cases we pick 3 of the flags and order the triple some way and then compute.

Lemma 7.1 *All 8 triple invariants associated to a pair of orthogonal flags have the same sign.*

Proof: At least for generic choices, we can normalize so that one of the pairs is given by

$$([r, 0, 1], [-1, 0, r]), \quad ([-1, 0, r], [r, 0, 1]),$$

and the other one is given by

$$([x, y, 1], [-x, -y, x^2 + y^2]) \quad ([-x, -y, x^2 + y^2], [x, y, 1])$$

We compute that the triple invariants occur in pairs. They are t_1 and t_2 and $1/t_1$ and $1/t_2$, where

$$t_1 = \frac{(r-x)(r(x^2+y^2)+x)}{(rx+1)(-rx+x^2+y^2)},$$

$$t_2 = \frac{(x^2+y^2+1)(-rx+x^2+y^2)(r(x^2+y^2)+x)}{(r-x)(rx+1)((x^2+y^2)^2+x^2+y^2)}.$$

The important thing to notice is that

$$\frac{t_1}{t_2} = \frac{(r-x)^2(x^2+y^2)}{(rx-x^2-y^2)^2} > 0.$$

Hence t_1 and t_2 have the same sign. Taking reciprocals does not change the sign, so all the triple products have the same sign. ♠

Here is the geometric picture. In the negative triple product case, the lines of the pair a_1, a_2 separate the points of the pair b_1, b_2 from each other, and *vice versa*. Figure 7.2 shows examples of the positive and the negative cases.

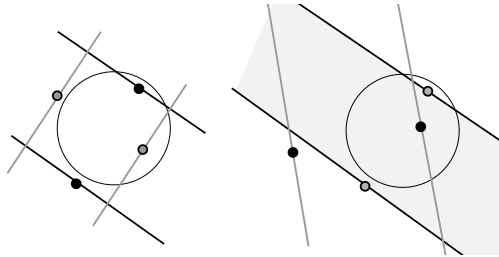


Figure 7.2: The positive case (left) and the negative case (right).

More generally, we say that two pairs of flags (a_1, a_2) and (b_1, b_2) are *separating* if the lines of the first pair separate the points of the second pair and *vice versa*.

7.3 The Embedding Proof

Transversality: Our embedding proof really just uses one familiar property of Anosov embeddings, namely *transversality*. An Anosov representation of a group G includes an equivariant map

$$\phi : \partial G \rightarrow \mathcal{F}, \tag{26}$$

where ∂G is the Gromov boundary of G – in our case a Cantor set – and \mathcal{F} is the flag variety. The key property is that every pair of flags of $\phi(\partial G)$ consists of transverse flags. Again, this means that the point of one flag does not lie in the line of the other. See e.g. [Lab] or [GW]. For reference below, we call this collection of flags the *big collection*.

Now we get to the proof. As we have already mentioned, each prism group preserves an infinite pattern of flats. These are just the orbit of the flats in the initial prism under the group. Moreover, each flat in the orbit has a distinguished geodesic, and so the pattern of geodesics is embedded provided that the pattern of flats is embedded.

In [S0] we proved that the pattern of flats is embedded when the group is a Pappus modular group. Our proof in the Anosov case is similar in spirit, but takes advantage of the transversality property discussed above. Corresponding to our prism representation ρ we have two infinite collections of flags, one subsuming the other. We have the big collection mentioned in the previous section. We also have the *small collection*. As in the previous chapters we make a choice of attracting over repelling. We then take the attracting flag for the element $g_\rho^2 = \rho(\sigma_2\sigma_3\sigma_2\sigma_3)$ and consider its orbit under the group. This is the small collection. The big collection contains the small collection.

Lemma 7.2 *Every triple of flags in the big collection has negative triple invariant.*

Proof: This is clearly true for the symmetric Pappus modular group. As we move continuously to other representations, the invariant cannot change sign without becoming 0 along the way. But if this happens, we have a non-transverse pair of flags, a contradiction. ♠

The flats in our pattern are naturally associated to the morphed marked boxes in the orbit. In the case of Pappus modular group, the flats are defined in terms of the tops and the bottoms of the marked boxes. As we move into the Anosov representations we define the same kind of association just by continuity. We call a pair of flags (a_1, a_2) *linked* if they are associated to the same morphed marked box. This is the same as saying that they are associated to the same flat in our pattern.

Lemma 7.3 (Separating) *Let (a_1, a_2) and (b_1, b_2) be pairs of linked flags. Then this pair is not separating.*

Proof: This also follows from transversality and continuity. The property is true for shears of the symmetric Pappus group, as one can see from the nesting of the geodesics in the hyperbolic plane associated to these groups. The general case follows from continuity. We we move along a path of Anosov representation, we can never acquire the separating property. If we did, we would encounter a non-transverse pair of flags. ♠

Now we will suppose that a pair of flats in our pattern intersect. We can move these flats by an isometry so that their intersection point is the origin. Now we have a linked pair of orthogonal flags. All the triple invariants associated to these flags must be negative. Hence the linked pair is separating. This contradicts the Separating Lemma. Hence the flats cannot intersect. This proves that our pattern of flats is embedded. This finishes the proof of Theorem 1.2. To finish the proof of Theorem 1.3 we just need to explain the shearing.

7.4 Shearing

Consider two representations (Π, p_1) and (Π, p_2) corresponding to points in the same ray. The crucial point is that p_1 and p_2 lie in the same medial geodesic γ in the same flat F of Π . Let $\delta_1 = \rho_1(\sigma_2)$ and $\delta_2 = \rho_2(\sigma_2)$ denote the elliptic polarities fixing p_1 and p_2 respectively. Let I be the isometry that translates along γ , mapping p_1 to p_2 . Another crucial point is that δ_k conjugates I to I^{-1} because δ_k stabilizes γ and reverses its directions. Hence

$$\delta_2 = I \circ \delta_1 \circ I^{-1} = I^2 \circ \delta_1.$$

In particular, if τ is the triangle in the pattern contained in the prism Π , then

$$\delta_2(\tau) = I^2(\delta_1(\tau)).$$

Thus the pair $(\tau, \delta_1(\tau))$ is replaced by the sheared pair $(\tau, I^2 \circ \delta_1(\tau))$. This is the shearing phenomenon in Theorems 1.2 and 1.3.

Finally, the rays are canonically parametrized as follows. We say that a point on a given ray γ is d away from the endpoint if the corresponding representation is obtained from the Pappus group at the endpoint by performing a shear of strength d . This pins down the shearing relationship.

It only remains to consider the strengths of the two kinds of shearing which produce the same group. It suffices to consider the generic case. We revisit the r, s, t coordinates from §5. If we fix the value of t , then the region $(r, s) \in (0, \infty)^2$ gives us all the prism groups (Π, p) with p varying in the same flat $F \cong \mathbf{R}^2$ of Π .

Let $r = \mu s$, as in Equation 19. The representation corresponding to (r, s, t) is a Pappus representation. According to Equation 19 and Lemma 5.5, the representation corresponding to $(rd, s/d, t)$ is a $\sqrt{2} \log d$ shearing of a Pappus representation. When $d > 1$ this is one kind of shearing and when $d < 1$ this is the other.

We compute that eigenvalues of our element g^2 are $-d^2, -d^{-2}, 1$, with the flag (b_1, L_2) corresponding to the d^2 eigenvalue. The eigenvalue set only depends on the shearing strength, namely $\sqrt{2} \log d$, and not any other property of the parameters. The two prism descriptions of the same representation give rise to the same element g^2 . Therefore they correspond to the same strength shears of both kinds. This completes the proof of Theorem 1.3.

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