

On projective evolutes of polygons

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Abstract

The evolute of a curve is the envelope of its normals. In this note we consider a projectively natural discrete analog of this construction: we define projective perpendicular bisectors of the sides of a polygon in the projective plane, and study the map that sends a polygon to the new polygon formed by the projective perpendicular bisectors of its sides. We consider this map acting on the moduli space of projective polygons.

We analyze the case of pentagons; the moduli space is 2-dimensional in this case. The second iteration of the map has one integral whose level curves are cubic curves, and the transformation on these level curves is conjugated to the map $x \mapsto -4x \pmod{1}$. We also present the results of an experimental study in the case of hexagons.

1 Introduction

Given a k -sided polygon P , we define the *projective normals* n_1, \dots, n_k by the construction shown in Figure 1 for $k = 5$. Figure 1 just shows the construction of n_1 but the other normals are constructed similarly.

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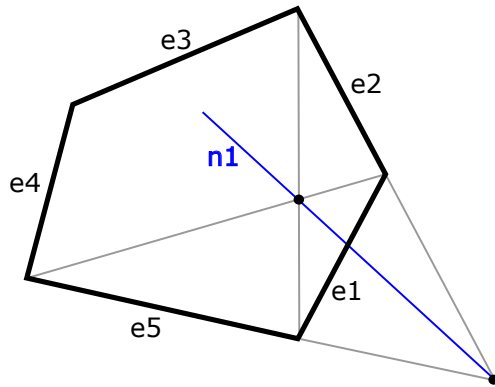


Figure 1: Constructing the projective normals

We get a new polygon $T(P)$ whose vertices are $n_1 \cap n_2, n_2 \cap n_3$, etc. Figure 2 shows an example.

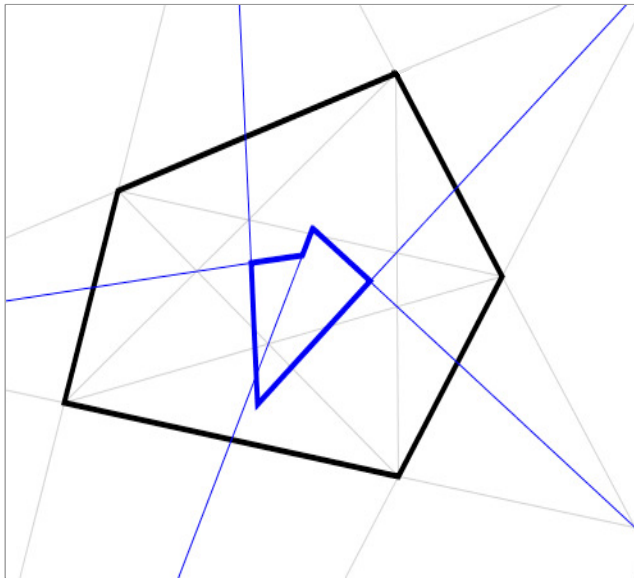


Figure 2: P in black and $T(P)$ in blue.

The map T is projectively natural, since it is defined entirely in terms of lines and their intersections. If P and Q are projectively equivalent polygons, then so are $T(P)$ and $T(Q)$. In particular, the map T is well defined on the moduli space M_k of projective equivalence classes of k -gons in the projective plane.

The case $k = 5$ is the first nontrivial case. It is specially attractive because M_5 is just 2 dimensional. On M_5 , the map T^2 has a nicer action than T . In

this note we will describe structural algebraic properties of T^2 on M_5 and also describe the dynamics. We work over the reals.

Theorem 1.1 *The map T^2 acts on $M_5(\mathbf{R})$ in such a way as to preserve a pencil of elliptic curves given by a single invariant rational function, I . Moreover, T^2 is conformal-symplectic in the sense that there is an area form ω on M_5 such that $(T^2)^*(\omega) = -4\omega$.*

See Equations 3 and 4 for I and ω respectively.

Theorem 1.2 *The map T^2 preserves each unbounded component of each invariant elliptic curve, and the restriction of T^2 to such a component, upon completion, is conjugate to the map $x \rightarrow -4x$ on the circle \mathbf{R}/\mathbf{Z} .*

By *unbounded* we mean that the component intersects the affine plane \mathbf{R}^2 in an unbounded set. The level sets all have one unbounded component and sometimes they have a bounded component as well. See Lemma 3.2 for a precise statement. When there is also a bounded component, T^2 maps the bounded component to the unbounded component. The bounded components consist of pentagons which are either convex or star-convex. See the remark at the end of §3.1. Figure 2 shows this phenomenon in action: P is convex and $T(P)$ is not. This situation explains how T^2 “blows up” around the regular pentagon. A nearly regular pentagon lies on a tiny bounded level set, and then T^2 stretches this tiny set all the way around the big unbounded component.

Our motivation for studying T is two-fold. On the one hand, in [1] two of us studied the dynamics of a related map defined in terms of the perpendicular bisectors of the sides of P . This Euclidean-geometry construction is a discrete analogue of the map that sends a smooth curve to its evolute. So, we view the map here as a projectively natural analogue of the discrete evolute map. On the other hand, in [5] one of us studied the map which sends the polygon P to the new polygon $P^\#$ whose vertices (referring to Figure 1) are the intersection points $n_1 \cap e_1, n_2 \cap e_2, \dots$. We called this map the *projective heat map* to bring out some analogy with discrete heat flow.

In §2 we prove Theorem 1.1. We first derive the equation for the map T in the most straightforward way. We then give a more general derivation which relates nicely to Frieze patterns and cluster algebras and explains the conformal symplectic nature of the map in conceptual terms. This second derivation is not needed for the proof of Theorem 1.2 however.

In §3 we prove Theorem 1.2. This amounts to an analysis of the pencil of elliptic curves and the geometry imposed on them by the pair (I, ω) .

In §4 we have a brief discussion of what we see for polygons with an even number of sides, concentrating on hexagons.

2 Algebraic Structure

2.1 A Formula for the Map

Let \mathbf{RP}^2 denote the real projective plane. The point $[a : b : c] \in \mathbf{RP}^2$ denotes the scale equivalence class of vectors (ra, rb, rc) with $r \in \mathbf{R} - \{0\}$. Dually, $[a : b : c]$ also represents the line given by $ax + by + cz = 0$. The cross product $(a_1, b_1, c_1) \times (a_2, b_2, c_2)$ naturally represents the line through $[a_1 : b_1 : c_1]$ and $[a_2 : b_2 : c_2]$. Dually, if these objects are interpreted as lines, then the cross product represents their intersection.

The non-singular linear transformations induce automorphisms of \mathbf{RP}^2 which map lines to lines. These automorphisms are called *projective transformations*. The projective transformations act simply transitively on the set of general position 4-tuples of points.

Each element of M_5 is uniquely projectively equivalent to one with vertices V_1, \dots, V_5 given by

$$[0 : -1 : 1], \quad [1 : 0 : 0], \quad [0 : 1 : 0], \quad [-1 : 0 : 1], \quad [x : y : 1]. \quad (1)$$

We call this equivalence class $P(x, y)$. Let

$$\begin{aligned} n(V_1, V_2, V_3, V_4) &= V'_1 \times V'_2, \\ V'_1 &= (V_1 \times V_3) \times (V_2 \times V_4), \\ V'_2 &= (V_1 \times V_2) \times (V_3 \times V_4). \end{aligned}$$

Then $n(V_1, V_2, V_3, V_4)$ gives the vector representing the projective normal line associated to the edge V_2V_3 of P . Let

$$W_1 = n(V_1, V_2, V_3, V_4), \quad W_2 = n(V_2, V_3, V_4, V_5), \quad \dots$$

$$X_1 = W_2 \times W_3, \quad X_2 = W_3 \times W_4, \quad \dots$$

The vectors X_1, \dots, X_5 represent the vertices of $T(P(x, y))$.

We normalize $T(P(x, y))$ as in Equation 1 to get $P(\bar{x}, \bar{y})$. We compute that

$$(\bar{x}, \bar{y}) = \left(\frac{(1+y)(1+x-xy)^2}{(1+x)(-1-y+xy)(1+x-y^2)}, \frac{(x-y)^2(1+x+y)}{(1+y-x^2)(1+x-y^2)} \right), \quad (2)$$

Our map is $T(x, y) = (\bar{x}, \bar{y})$.

2.2 The Invariants

Some members of M_5 are degenerate, namely the ones which have triples of collinear points. In terms of our coordinates, this happens for the line at infinity and for the lines

$$x + 1 = 0, \quad y + 1 = 0, \quad x + y + 1 = 0, \quad x = 0, \quad y = 0.$$

It turns out that a certain product of these defining equations is an invariant for the map T^2 . Define

$$I(x, y) = \frac{(x+1)(y+1)(x+y+1)}{xy}. \quad (3)$$

A direct calculation in Mathematica shows that

$$I(x, y)I(\bar{x}, \bar{y}) = -1.$$

Hence $I \circ T^2 = I$. This is our invariant.

The conformally invariant area form is given by

$$\omega = \frac{1}{xy} dx \wedge dy. \quad (4)$$

To verify this, we let J denote the Jacobian of T^2 . We compute that

$$\frac{J(x, y)}{\bar{x}\bar{y}} = \frac{-4}{xy}.$$

This is equivalent to the statement that $(T^2)^*(\omega) = -4\omega$.

This completes the proof of Theorem 1.1.

2.3 A Different Derivation

In this section we derive the equation for T in a different way. This derivation is more elaborate, but it has two advantages. First, it generalizes more nicely to polygons with more sides. Second, the derivation puts into perspective the invariant quantities from Theorem 1.1, relating them to topics such as the pentagram map and cluster algebras. This material is not needed for the proof of Theorem 1.2.

It is convenient to work in \mathbf{R}^3 . An n -gon in the projective plane can be lifted to a polygon in \mathbf{R}^3 . Such a lifting is not unique, but if n is not a multiple of 3, we can normalize the lifting by requiring that the determinant of every triple of its consecutive vertices equals 1, and this makes this lifting unique (cf. [4], Proposition 4.1). We call the polygons satisfying this determinant relation *unimodular*.

Let P_1, \dots, P_n be the vertices of the lifted unimodular n -gon. Since

$$\det(P_{i-1}, P_i, P_{i+1}) = 1$$

for all i , we have

$$P_{i+2} = a_{i+1}P_{i+1} - b_iP_i + P_{i-1}, \quad (5)$$

where a_i, b_i are two n -periodic sequences. These coordinates, a_i, b_i , are invariant under the diagonal action of $SL(3, \mathbf{R})$ on polygons. The formulas for the map given above are entirely in terms of cross products, so it make sense to apply it to unimodular polygons. For the sake of getting the indices correct, let us write it out again, using (P, Q) in place of (V, X) . We make this change because the indices here are slightly different than the ones given above.

$$\begin{aligned} Q_i &= [((P_{i-2} \times P_{i-1}) \times (P_i \times P_{i+1})) \times ((P_{i-2} \times P_i) \times (P_{i-1} \times P_{i+1}))] \\ &\quad \times [((P_{i-1} \times P_i) \times (P_{i+1} \times P_{i+2})) \times ((P_{i-1} \times P_{i+1}) \times (P_i \times P_{i+2}))], \quad (6) \end{aligned}$$

From now on, we specialize to the case $n = 5$. In particular, we take indices mod 5. Analogs of the three lemmas that follow exist for other values of n not divisible by 3.

Since M_5 is two-dimensional, the 10 coefficients a_i, b_i , $i = 1, \dots, 5$, depend on two parameters, as in Example 5.6 of [4].

Lemma 2.1 *We have $b_i = a_{i+3}$, $a_i + 1 = a_{i+2}a_{i+3}$.*

Proof: Equation (5) implies

$$a_{i+1} = \det(P_{i-1}, P_i, P_{i+2}), \quad b_i = \det(P_{i-1}, P_{i+1}, P_{i+2}),$$

therefore $b_i = a_{i+3}$. Also

$$P_{i+3} = (a_{i+2}a_{i+1} - b_{i+1})P_{i+1} + (1 - a_{i+2}b_i)P_i + a_{i+2}P_{i-1}.$$

Since $\det(P_{i+3}, P_{i-1}, P_i) = 1$, we conclude that $a_{i+2}a_{i+1} - b_{i+1} = 1$, therefore $a_i + 1 = a_{i+2}a_{i+3}$. ♠

Set $a_3 = x, a_1 = y$, then

$$a_4 = \frac{1+y}{x}, \quad a_2 = \frac{1+x+y}{xy}, \quad a_5 = \frac{1+x}{y}.$$

The coordinates x, y determine the projective equivalence class of a pentagon.

The numbers a_i comprise the rows of a frieze pattern

$$\begin{array}{cccccccc} 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\ & x & & \frac{y+1}{x} & & \frac{x+1}{y} & & y & & \frac{x+y+1}{xy} & x \\ & & y & & \frac{x+y+1}{xy} & & x & & \frac{y+1}{x} & & \frac{x+1}{y} & & y \\ 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \end{array}$$

related to the Pentagramma Mirificum of Gauss, see [3].

Let $\{U_i\}$ be a (not necessarily unimodular) pentagon in \mathbf{R}^3 . Let $Q_i = t_i U_i$ be a rescaling, such that the pentagon Q is unimodular. Set $D_i = \det(U_{i-1}, U_i, U_{i+1})$.

Lemma 2.2 *One has*

$$t_i = \frac{(\prod_i D_i)^{1/3}}{D_{i-1}D_{i+1}}.$$

Proof: One needs to solve the system of five equations

$$t_{i-1}t_it_{i+1} = \frac{1}{D_i}, \quad i = 1, \dots, 5,$$

which becomes a linear system after taking logarithms. Its solution is as stated. ♠

The unimodular pentagon Q satisfies the recurrences

$$Q_{i+2} = \bar{a}_{i+1}Q_{i+1} - \bar{b}_iQ_i + Q_{i-1},$$

where the coefficients satisfy the conditions of Lemma 2.1.

Lemma 2.3 *One has*

$$\bar{a}_{i+1} = \frac{\det(U_{i-1}, U_i, U_{i+2})}{\det(U_{i-1}, U_i, U_{i+1})}.$$

Proof: Since

$$\bar{a}_{i+1} = \det(Q_{i-1}, Q_i, Q_{i+2}) = t_{i-1}t_it_{i+2} \det(U_{i-1}, U_i, U_{i+2}),$$

the result follows by substituting the values of t_i from Lemma 2.2. ♠

Let \bar{x} and \bar{y} denote the respective variables related to \bar{a}_i and \bar{b}_i as in Lemma 2.1. We again write our map as $T(x, y) = (\bar{x}, \bar{y})$. A Mathematica calculation using formula (6) and Lemma 2.3 yields the same equation for T as we got in Equation 2.

This alternate derivation puts the invariant quantities in perspective. The integral I equals $\prod_i a_i$. The product $\prod_i a_i$ is a monodromy integral of the pentagram map, see Example 5.6 in [4]. Curiously, we also can write

$$I = \sum_i a_i + 3.$$

This alternate form can be deduced from the relations from Lemma 2.1.

The symplectic form ω is known in the theory of cluster algebras; the spaces of frieze patterns of arbitrary width possess analogous (pre)symplectic structures. The function I and the form ω appeared in the study of the cross-ratio dynamics on ideal polygons in [2]: in contrast with Theorem 1.1, both are invariant under the cross-ratio dynamics in the case of ideal pentagons. See Section 7.1.3 of [2].

3 The Dynamics

In this section we prove Theorem 1.2.

3.1 The Invariant Curves

For each real r , the map T^2 preserves the curve $I(x, y) = r$. The equation for this curve is

$$(x+1)(y+1)(x+y+1) - rxy = 0. \tag{7}$$

This is an example of an elliptic curve. To understand it better, we homogenize the curve and consider it as a projective variety in \mathbf{RP}^2 . Homogenizing Equation 7 we get:

$$Q(x, y, z) = x^2y + xy^2 + x^2z + y^2z + (3 - r)xyz + 2xz^2 + 2yz^2 + z^3. \quad (8)$$

Lemma 3.1 *The elliptic curve in Equation 8 is nonsingular if $r \neq 0$ and $r \neq (11 \pm 5\sqrt{5})/2$.*

Proof: We consider the gradient. When $z = 0$ we have

$$\nabla Q = (2xy + y^2, 2xy + x^2, (3 - r)xy + x^2 + y^2).$$

Suppose $\nabla Q = 0$. If $y = 0$ then the second coordinate is x^2 , which forces $x = 0$. Now assume that $y \neq 0$. Setting the first coordinate equal to 0, we get $x = -y/2$. But then the second coordinate is $-3y^2/4$. This gives $x = y = 0$. So, when $z = 0$ we have no singular points at all.

When $z \neq 0$ it suffices to set $z = 1$ and consider the gradient (Q_x, Q_y) of the inhomogeneous equation. When $x = 0$ we have $Q_y = 2 + 2y$. This vanishes only when $y = -1$. But then $Q_x = r$. This only vanishes if $r = 0$. If $x = -1$ we have $Q_y = r$. Again this vanishes only if $r = 0$.

Let $\text{res}(Q_x, Q, y)$ denote the resultant of Q_x and Q with respect to y . Let

$$R_1 = \text{res}(Q_x, Q, y), \quad R_2 = \text{res}(Q_y, Q, y).$$

Since we have already analyzed the case $x = -1$, we can assume $x + 1 \neq 0$. It turns out that $x + 1$ divides R_1 and R_2 , so we divide out by $x + 1$ and compute

$$\text{res}(R_1/(x + 1), R_2/(x + 1), x) = -r^8(r^2 - 11r - 1)^2.$$

This only vanishes when r has one of the advertised values. ♠

Let E_r be the level curve corresponding to the invariant $I(x, y) = r$. Let

$$r_{\pm} = \frac{11 \pm 5\sqrt{5}}{2}.$$

Here $r_- \approx -0.09$ and $r_+ \approx 11.09$. Let $\mathbf{R}' = \mathbf{R} - \{0, r_-, r_+\}$. The set $\{-1, -0.05, 1, 12\}$ intersects each connected component of \mathbf{R}' . Figure 3 shows plots of E_r for r in this set.

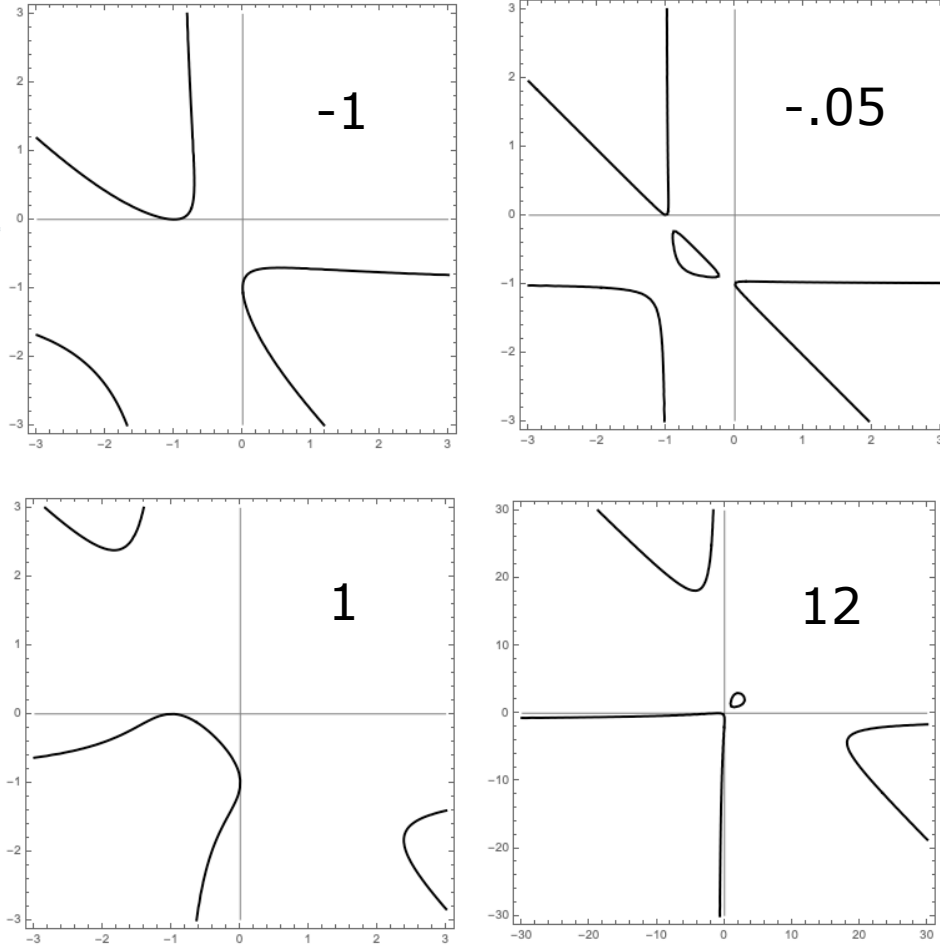


Figure 3: E_r for $r = -1, -0.05, 1, 12$.

Lemma 3.2 For all $r \in \mathbf{R}'$ the curve E_r has an unbounded component which contains the points

$$[1 : 0 : 0], [0 : 1 : 0], [1 : -1, 0], [-1 : 0 : 1], [0 : -1 : 1]$$

and which is otherwise disjoint from the coordinate axes and the line at infinity. When $r \in (r_-, 0)$ the curve E_r also has a bounded component that lies in the $(-, -)$ quadrant. When $r \in (r_+, \infty)$ the curve E_r also has a bounded component that lies in the $(+, +)$ quadrant.

Proof: We set $(-1, 0) = [-1 : 0 : 1]$ and $(0, -1) = [0 : -1 : 1]$ for ease of notation.

We have $Q(x, y, 0) = xy(x + y)$, so E_r intersects the line at infinity at the three points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ and $[1 : -1 : 0]$. Now, the topological type of E_r cannot change, as a function of r , unless r passes through a value where the curve is singular. Thus, the topological type does not change within each of the 4 intervals of \mathbf{R}' . We check, making an explicit plot for each of these points, that the topology is as stated. Hence, it is always as stated. See Figure 3.

We find that $Q(0, y, 1) = (1 + y)^2$. Hence $Q(0, y, 1) = 0$ if and only if $y = -1$. This means that our level sets intersect the x -axis only at $(-1, 0)$. A similar argument establishes this result for the y -axis. When only the unbounded components exist, they contain the points $(-1, 0)$ and $(0, -1)$. As r crosses into the regions which have bounded components, these components appear at points that do not lie on the coordinate axes. So, at least for some values in $(r_-, 0)$ and (r_+, ∞) , it is the unbounded components that contain these special points. But then the bounded components are always contained in single quadrants. Two evaluations are sufficient to check that the components are in the quadrants as stated. ♠

Remark: We reiterate what we said in introduction. The convex and star-convex pentagon classes lie on the bounded components, and conversely the bounded components consists of convex or star convex pentagon classes. Thus, the unbounded components consist of projective classes of pentagons which are neither convex nor star convex.

3.2 Intrinsic Boundedness

Let E be one of our elliptic curve level sets. Let X_I denote the Hamiltonian vector field with respect to the invariant I and the area form ω . We get X_I by rotating ∇I by 90 degrees counterclockwise and then multiplying both components by xy . That is

$$X_I = \left(\frac{(1+x)(1+x-y^2)}{y}, \frac{(1+y)(-1-y+x^2)}{x} \right)$$

The vector field X_I is tangent to the level curves. If X_I is entirely defined on some arc of a level set, X_I defines a metric on this arc. The distance between points on the arc is the time it takes to flow from one point to the other along X_I . More precisely, this defines a metric on all points of each

nonsingular level curve away from the points $(-1, 0)$ and $(0, -1)$, which are the only points where the level curves intersect the coordinate axes.

Each bounded component B is disjoint from the coordinate axes and ∇I is nonzero at all points of B . (This follows from the quotient rule and from the non-singularity of B .) But then X_I is entirely defined and nonzero on B . Hence B is isometric to $\mathbf{R}/\lambda\mathbf{Z}$ for some λ that depends on the level set.

Now let us consider some unbounded component U . The vector field X_I is defined and nonzero at all points of $U \cap \mathbf{R}^2$ except $(-1, 0)$ and $(0, -1)$. Since U has 3 points at infinity, our construction gives us a metric on U away from 5 points. We show that this metric is bounded, so that the completion is again isometric to $\mathbf{R}/\lambda\mathbf{Z}$ for some λ that depends on the parameter. We treat the points in turn.

Case 1: Consider the picture near $(-1, 0)$. We are going to restrict X_I to U and see what happens as we approach $(-1, 0)$. The x -axis is tangent to U at $(-1, 0)$ and also intersects U at the point $[1 : 0 : 0]$. Since the x -axis can only intersect U three times, counting multiplicity, we see that U cannot have an inflection point at $(0, 0)$. So, we may write $x = u - 1$ and $y = \alpha u^2 + \beta(u)u^3$. Here α is a nonzero constant and β is a function that remains bounded as $u \rightarrow 0$. With these substitutions, we find that

$$X_I \cdot X_I = \frac{1}{(u-1)^2(\alpha + \beta u)^2} \times (1 - 2u + O(u^2)).$$

But this means that $\|X_I\| \rightarrow 1/|\alpha|$ as $u \rightarrow 0$.

Case 2: The argument for $(0, -1)$ is the same as Case 1.

Case 3: Consider the picture near the point $[1 : 0 : 0]$. If we stay on the level set E_r we have $x \rightarrow \infty$ and $y \rightarrow -1$. We have

$$X_I \cdot X_I = \frac{x^6 + P(x, y)}{x^2 y^2},$$

where $P(x, y)$ is a polynomial whose largest degree in x is 5. Therefore, as we approach $[1 : 0 : 0]$ along E_r , we have $\|X_I\| \sim x^2$ along E_r . Starting near the point $(n, -1)$ we reach a point near $(n + 1, -1)$ in $1/n^2$ units of time. Since $\sum 1/n^2$ is a convergent series, we reach $[1 : 0 : 0]$ by flowing along X_I for a finite time.

Case 4: The argument for $[0 : 1 : 0]$ is the same as Case 3.

Case 5: Consider the picture near the point $[1 : -1 : 0]$. If we stay on the level set E_r we have $x + y + 1 \rightarrow r$. This time we have $|x|/|y| \rightarrow 1$ as we approach $[1 : -1 : 0]$. We have

$$X_I \cdot X_I = \frac{2x^4y^4 + P(x, y)}{x^2y^2},$$

where P is a polynomial whose monomials have maximum degree 7. From this we see that again $\|X_I\| \sim x^2$ as we approach $[1 : -1 : 0]$ along E_r . The same analysis as in Case 3 works here.

This completes the analysis. Now we know that each component of E_r has a metric completion which is isometric to $\mathbf{R}/\lambda\mathbf{Z}$ for some constant λ that depends on the value of r . In case E_r works for both components, we guess that the same λ works for both but we don't know how to prove this.

3.3 The Dynamics

We will prove Theorem 1.2 with respect to the space $\mathbf{R}/\lambda\mathbf{Z}$. The final conjugacy to \mathbf{R}/\mathbf{Z} is given by a similarity.

We first consider the cases when $r \in (-\infty, r_-) \cup (0, r_+)$. In this case, there is only the unbounded component to worry about. The vector field X_I gives a metric to E_r which (upon completion) makes it isometric to $\mathbf{R}/\lambda\mathbf{Z}$. The map T^2 preserves the level sets and multiplies the area form by -4 . From this we see that the differential $d(T^2)$ maps X_I to $-4X_I$.

Let $\psi : E_r \rightarrow \mathbf{R}/\lambda\mathbf{Z}$ be an isometry. Consider the conjugate map

$$\tau_2 = \psi \circ T^2 \circ \psi^{-1} : \mathbf{R}/\lambda\mathbf{Z} \rightarrow \mathbf{R}/\lambda\mathbf{Z}.$$

From what we have just said, τ_2 acts as multiplication by -4 wherever it is defined. Moreover, τ_2 is defined on all but finitely many points of $\mathbf{R}/\lambda\mathbf{Z}$.

The subset of $\mathbf{R}/\lambda\mathbf{Z}$ where τ_2 is defined is not connected; it consists of a finite number of intervals. On each interval τ_2 acts as multiplication by -4 . We want to see that τ_2 is continuous across these undefined points. It is more convenient to show that T^2 is continuous across the points where it is not defined. This is the same thing.

Let ξ be some point in E_r where T^2 is not defined. Let $J \subset E_r$ be some small interval containing ξ such that T^2 is entirely defined on $J - \{\xi\}$. Let J_1, J_2 be the two components of $J - \{\xi\}$. Restricting to J_j for each $j = 1, 2$ we get a limiting value

$$\zeta_j = \lim_{\xi' \in J_j \rightarrow \xi} T^2(\xi') \in E_r.$$

This follows from the fact that the restriction of T^2 to J_j is 4-Lipshitz.

Lemma 3.3 *We have $\zeta_1 = \zeta_2$.*

Proof: We will suppose that $\zeta_1 \neq \zeta_2$ and we will derive a contradiction. The idea is to work in local coordinates and hit the problem with some complex analysis. Let $\pi_1 : \mathbf{R}^2 \rightarrow \mathbf{R}$ be projection onto the first coordinate. We choose real projective transformations Ψ_1 and Ψ_2 such that

1. $\Psi_1(\xi) = (0, 0)$ and $\Psi_1(E_r)$ is tangent to the x -axis at $(0, 0)$.
2. $\Psi_2 \circ T^2(J_1 \cup J_2)$ is contained in compact subset of \mathbf{R}^2 .
3. $\pi_1 \circ \Psi_2(\zeta_1) \neq \pi_1 \circ \Psi_2(\zeta_2)$.

The second property uses the fact that the limits ζ_1 and ζ_2 exist.

If we choose J small enough there is an algebraic (and hence analytic) parametrization $\phi : (-\epsilon, \epsilon) \rightarrow \Psi_1(J)$ which is the inverse of π_1 . We can write $\phi(x) = (x, \phi_2(x))$ where ϕ_2 is an analytic function of one variable.

$$f = \pi_1 \circ \Psi_2 \circ T^2 \circ \Psi_1^{-1} \circ \phi.$$

By construction, f is discontinuous across 0. When we work over the complex numbers, the restriction of π_1 to a neighborhood of 0 in $\Psi_1(E_r)$ is a nonsingular holomorphic map. But then ϕ_2 is holomorphic in a neighborhood of 0 in \mathbf{C} . In particular, ϕ_2 has a convergent power series in a neighborhood of 0.

Continuing to work over the complex numbers, we have

$$f(z) = \frac{P(z, \phi_2(z))}{Q(z, \phi_2(z))} = \frac{p_k z^k + p_{k+1} z^{k+1} + \dots}{q_\ell z^\ell + q_{\ell+1} z^{\ell+1} + \dots} = z^{k-\ell} h(z). \quad (9)$$

Here P and Q are polynomials in 2 variables. Let us explain the rest of Equation 9. Since ϕ_2 has a convergent power series in a neighborhood of 0, the

functions $z \rightarrow P(z, \phi_2(z))$ and $z \rightarrow Q(z, \phi_2(z))$ also have convergent power series near 0, as we have written. The quotient of these two series has the given form, with h being a holomorphic function defined in a neighborhood of 0. If $k - \ell < 0$ then the restriction of f to $(0, \epsilon)$ would be unbounded. This contradicts Item 2 above. Hence f has a removable singularity at 0. In particular, f extends continuously to 0. This is a contradiction. ♠

This argument works for any missing point of E_r . We conclude that τ_2 is globally the map $x \rightarrow -4x$ on $\mathbf{R}/\lambda\mathbf{Z}$.

It remains to consider the cases when $r \in (r_-, 0) \cup (r_+, \infty)$. We will consider the case when $r \in (r_+, \infty)$. The other case has the same treatment. A single evaluation suffices to show that T^2 maps the bounded component to the unbounded component. For instance $I(3, 4) = 40/3$ and this point lies on the bounded component. We compute that $T^2(3, 4)$ and $T^4(3, 4)$ both lie in the $(-, +)$ quadrant. Hence both these points lie on the unbounded component. Thus, T^2 maps both the bounded and unbounded components to the unbounded component. Dynamically, we could say that a pentagon loses convexity (or star-convexity) immediately when the map is applied.

This completes the proof of Theorem 1.2.

4 Polygons with More Sides

Here we briefly discuss some things we observed for polygons with an even number of sides. We say that a $2n$ -gon is *axis aligned* if its sides are alternately horizontal and vertical. Let Ω_{2n} denote the set of these. It is not hard to see that $T(\Omega_{2n}) = \Omega_{2n}$. If the k th side of $P \in \Omega_{2n}$ is vertical (respectively horizontal) then the k th side of $T(P)$ is horizontal (respectively vertical). For this reason, it makes good sense to reflect in the diagonal line $y = x$ after applying T . The simplest conjecture is that Ω_{2n} is a global attractor for T . This definitely appears to be the case for Ω_6 and we have some numerical evidence that this is also true for Ω_8 . We hope to return to these kinds of results in a later paper.

We first explain how Ω_6 embeds in M_6 . Letting (V_1, \dots, V_6) be a hexagon, we normalize so that V_1, \dots, V_6 are given by

$$(0, 1), \quad (-1, 1), \quad (-1, 0), \quad (0, 0), \quad (x_5, y_5), \quad (x_6, y_6)$$

The coordinates (x_5, y_5, x_6, y_6) are coordinates for M_6 .

We define

$$A = x_5 + x_6 + 1, \quad B = x_5 - x_6 + 2y_5 - 1, \quad C = 2y_5 - 1, \quad D = y_6 - y_5.$$

The set of equivalence classes in M_6 which are represented by elements of Ω_6 is given by

$$A^2 - B^2 + C^2 = 1, \quad D = 0.$$

Now we discuss the dynamics of T on Ω_6 . For this purpose it is convenient to change coordinates. We normalize a hexagon in Ω_6 to have vertices

$$(0, 0), \quad (a, 0), \quad (a, b), \quad (1, b), \quad (1, 1), \quad (0, 1).$$

We call this hexagon $H(a, b)$. We then apply T , then reflect in the diagonal, then apply an affine transformation which preserves the vertical and horizontal directions and carries the hexagon back to the same form. The new hexagon has the equation $H(f(a), f(b))$ where

$$f(t) = \frac{2t - 1}{t^2 - 1}, \quad (10)$$

The map f is a degree 2 expanding map from $\mathbf{R} \cup \infty$ to itself.

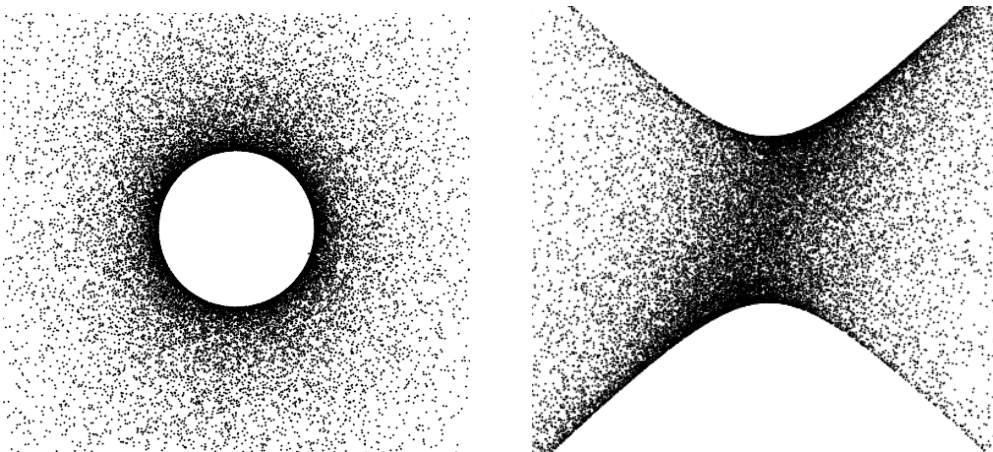


Figure 4: The orbit of a hexagon projected on the (A, C) - and the (B, C) -planes, respectively.

We think that almost every orbit of the map $(a, b) \rightarrow (f(a), f(b))$ has dense orbits but we did not work out a proof. In short, it appears that for hexagons, everything in M_6 is attracted to the image of Ω_6 in M_6 and then (after changing coordinates) the map on Ω is given by $(a, b) \rightarrow (f(a), f(b))$.

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