A Trichotomy for Rectangles Inscribed in Jordan Loops

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Abstract

We prove a general structural theorem about rectangles inscribed in Jordan loops. One corollary is that all but at most 4 points of any Jordan loop are vertices of inscribed rectangles. Another corollary is that a Jordan loop has an inscribed rectangle of every aspect ratio provided it has 3 points which are not vertices of inscribed rectangles.

1 Introduction

A *Jordan loop* is the image of the circle under a continuous injective map into the plane. O. Toeplitz conjectured in 1911 that every Jordan loop contains 4 points which are the vertices of a square. This is often called the *Square Peg Conjecture.* An affirmative answer is known in many special cases. In 1913, Emch [**Emch**] proved the result for convex curves. In 1944, L. G. Shnirlmann [**Shn**] proved the result for sufficiently smooth curves. In 1961, R. Jerrald [**Jer**] extended this to the case of C^1 curves. Recently T. Tao [**Ta**] proved the result for special curves having even lower regularity. The above is a very partial survey of the literature. The 2014 survey paper by B. Matschke [**Ma1**] and the recent book by I. Pak [**P**] have extensive discussions of the history of the Square Peg Conjecture and many additional references.

There is also some work done in the case of rectangles. In 1977, H. Vaughan [**Va**] gave a proof that every Jordan loop has an inscribed rectangle.

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A recent paper of C. Hugelmeyer [**H**] combines Vaughan's basic idea with some very modern knot theory results to show that a smooth Jordan loop always has an inscribed rectangle of aspect ratio $\sqrt{3}$. The recent paper [**AA**] proves that any quadrilateral inscribed in a circle can (up to similarity) be inscribed in any convex smooth curve. See also [**Ma2**]. In the recent paper [**ACFSST**], the authors show that every Jordan Loop contains a dense set of points which are vertices of inscribed rectangles. For additional work on inscribed rectangles, see [**Mak1**], [**Mak2**], and [**MW**].

Relatedly, one can consider the situation for triangles. In 1980, M. Meyerson [**M**] proved that all but at most 2 points of any Jordan loop are vertices of inscribed equilateral triangles. This result is sharp because two points of a suitable isosceles triangle are not vertices of inscribed equilateral triangles. In 1992, M. Neilson [**N**] proved that an arbitrary Jordan loop contains a dense set of points which are vertices of inscribed triangles of any given shape.

We are going to prove a strong version of Meyerson's Theorem for rectangles. Let $I(\gamma)$ denote the space of all labeled rectangles inscribed in γ. We always label a rectangle R so that the vertices of R go counterclockwise around R. We orient γ so that it goes counterclockwise around the region in the plane it bounds. The space $I(\gamma)$ is naturally a subset of $\boldsymbol{R}^8 = (\boldsymbol{R}^2)^4$.

We call a rectangle R inscribed in γ *graceful* if the cyclic order imparted on the vertices of R by the ordering on γ coincides with the ordering we have already given to the vertices of R. Let $G(\gamma) \subset I(\gamma)$ denote the space of gracefully inscribed labeled rectangles.

The *aspect ratio* of a rectangle in $I(\gamma)$ is the length of the second side divided by the length of the first side. Given $S \subset I(\gamma)$, let $V(\gamma, S) \subset \gamma$ denote the subset of γ consisting of points which are vertices of rectangles in S. Let $\rho(S)$ denote the union of all the aspect ratios of rectangles in S.

Theorem 1.1 (Trichotomy) *Let γ be an arbitrary Jordan loop. Then $I(\gamma)$ contains a connected set S satisfying one of the following.*

1. *Members of S have uniformly large area, $V(\gamma, S) = \gamma$, and $1 \in \rho(S)$.*
2. *Members S have uniformly large diameter, $\rho(S) = (0, \infty)$, and $V(\gamma, S)$ contains all but at most 4 points of γ.*
3. *The set S has members of every sufficiently small diameter, and $V(\gamma, S)$ contains all but at most 2 points of γ.*

Moreover, $S \subset G(\gamma)$.

We mention three corollaries. The first two corollaries are immediate, and we will prove the third one in §2.7.

Corollary 1.2 *Let γ be any Jordan loop. Then all but at most 4 points of γ are vertices of rectangles gracefully inscribed in γ.*

This result is sharp: There are 4 points of a non-circular ellipse which are not vertices of any inscribed rectangle.

Corollary 1.3 *Let γ be any Jordan loop. If γ has 3 points which are not vertices of gracefully inscribed rectangles then γ has gracefully inscribed rectangles of every aspect ratio.*

Corollary 1.4 *Let γ be any Jordan loop and let μ be any non-atomic measure on γ having mass 1. Then γ has a gracefully inscribed rectangle γ such that the total μ-measure of each pair opposite sides of γ cut off by R is $1/2$.*

Remarks:
(1) I describe the cases in the Trichotomy Theorem respectively as *elliptic*, *hyperbolic*, and *parabolic*, because the geometry of the situation seems to vaguely resemble the action of these kinds of linear transformations on $\boldsymbol{R}^2$.
(2) Note that there are examples, such as the circle, for which both the hyperbolic and elliptic cases occur.
(3) I conjecture that the parabolic case cannot actually occur. This conjecture immediately implies the Square Peg Conjecture.
(4) The elliptic case occurs for any curve with 4-fold rotational symmetry but I conjecture that the hyperbolic case also occurs for every Jordan loop. This conjecture implies the much stronger result that every Jordan loop has an inscribed rectangle of every aspect ratio – a conjecture that is not even known in the smooth or polygonal cases.
(5) One could view Corollary 1.2 as a version of Meyerson's Theorem [**M**] for rectangles, though our proof is much different.
(6) The paper [**ACFSST**] also has Corollary 1.4 (without the graceful bit) when γ is rectifiable and μ is arc-length measure normalized to have total length 1. The methods are different.

We prove the Trichotomy Theorem by taking a suitable limit of the polygonal case. Let γ be a polygon. By an *arc component* of $I(\gamma)$ we mean a connected component of $I(\gamma)$ which is homeomorphic to an arc. By *proper*,

we mean that as one moves towards an endpoint of an arc component in $I(\gamma)$, the aspect ratio tends either to 0 or to ∞. Moreover, we insist that the rectangles at each end of a proper arc accumulate on a chord of γ and that the two chords are distinct. The left side of Figure 1 below suggests an example of a proper arc in $I(\gamma)$ when γ is an equilateral triangle. The black segments are the two chords of accumulation. This arc actually is a component of $G(\gamma)$.

Theorem 1.5 *There is an open dense subset $\mathcal{P}$ of polygons with the following property. For each $\gamma \in \mathcal{P}$ the space $I(\gamma)$ is a piecewise smooth 1-manifold whose arc components are proper. Moreover the aspect ratio function $\rho : I(\gamma) \to (0, \infty)$ is injective in a neighborhood of each smooth point of $I(\gamma)$ and, $\rho^{-1}(1)$ consists entirely of smooth points.*

We define 2 kinds of components of $I(\gamma)$.

- A component A of $I(\gamma)$ is a *hyperbolic component* if the aspect ratio of the rectangles in A tends to 0 as the rectangles tend towards one endpoint of A, and to ∞ as they tend to the other endpoint of A. The example in Figure 1 is a hyperbolic component.

- The operation of cyclically relabeling gives a $\boldsymbol{Z}/4$ action on the space $I(\gamma)$ which has no fixed points. We call a component of $I(\gamma)$ *elliptic* if it is stabilized by the $\boldsymbol{Z}/4$ action. These components are loops.

We call a component of $I(\gamma)$ *global* if it is either hyperbolic or elliptic. The reason for the name is that $V(\gamma, S)$ contains all but at most 4 points of γ when γ is hyperbolic (Lemma 2.3) and all points of γ when γ is elliptic (Lemma 2.8).

The left and right hand sides of Figure 1 respectively suggest a hyperbolic and an elliptic component. The figure on the right, drawn by hand, may have small inaccuracies. The basic construction is to have the centers of the rectangles wind once around a small central curve, while the rectangles themselves rotate one quarter of the way around and return to their original location but with the corresponding cyclic relabeling. When this operation is repeated 4 times, one has an elliptic component. It would be simpler to keep the centers fixed, but I wanted to show an example without 4 fold rotational symmetry.

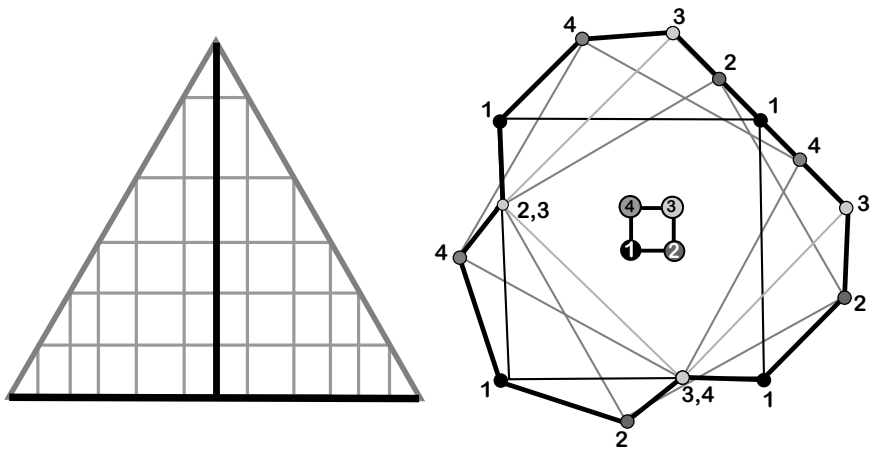

Figure 1: A hyperbolic component and an elliptic component.

The relabeling action permutes the various components of $I(\gamma)$ and we call the orbits of this action the *unlabeled components*. We define the following quantities:

- $\Omega(\gamma)$ is the number of unlabelled inscribed squares.
- $\Omega_H(\gamma)$ is the number of unlabelled hyperbolic components.
- $\Omega_E(\gamma)$ is the number of unlabeled elliptic components.

We will establish the following equation for each $\gamma \in \mathcal{P}$.

$$\Omega(\gamma) + \Omega_H(\gamma) + \Omega_E(\gamma) \equiv 0 \bmod 2. \tag{1}$$

It is well known that for the generic polygon the number of unlabeled inscribed squares is odd. See for instance [**St**] or [**P**, Theorem 23.11]. Hence $I(\gamma)$ always contains a global component. Finally we prove the following result.

Theorem 1.6 *For $\gamma \in \mathcal{P}$ the only global components of $I(\gamma)$ belong to $G(\gamma)$.*

Theorem 1.6 now implies the following result.

Theorem 1.7 *For each $\gamma \in \mathcal{P}$ the space $G(\gamma)$ contains a global component.*

We get the Trichotomy Theorem by taking a suitable limit of Theorem 1.7.

We also mention another corollary of Theorem 1.6: A generic polygon has an odd number of gracefully inscribed squares. See §3.4. I don't think that this corollary follows directly from [**P**, Theorem 23.11], which makes a statement about the parity of the number of all inscribed squares.

Here is an outline of the paper. In §2 we will deduce the Trichotomy Theorem from Theorem 1.7. Following §2, the rest of the paper is about polygons.

In §3 we will deduce Equation 1 from Theorem 1.5.

In §4 we prove Theorem 1.5. This is really just an exercise in transversality, and many methods would work.

In §5 we prove Theorem 1.6.

We warn the reader about one persistent abuse of terminology. When we speak of *a rectangle in* $I(\gamma)$ (or in related configuration spaces) we mean the rectangle corresponding to a member of $I(\gamma)$ and not some kind of configuration of 4 elements of $I(\gamma)$. We hope that this does not cause confusion.

One thing I would like to mention is that I discovered all the results in this paper by computer experimentation. I wrote a Java program which computes the space $G(\gamma)$ in an efficient way for polygonal loops γ having up to about 20 sides.

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2 The Trichotomy Theorem

In this chapter we deduce the Trichotomy Theorem from Theorem 1.7 using a limiting argument. We begin with some preliminary material on point-set topology.

2.1 Hausdorff Limits

Suppose that C is a compact metric space. Let X_C denote the set of closed subsets of C. We define the *Hausdorff distance* between closed $A, B \subset C$ to be the infimal ϵ such that each of the two sets is contained in the ϵ-tubular neighborhood of the other one. This definition makes X_C into a compact metric space.

Lemma 2.1 *Let $\{A_n\}$ be a sequence of nonempty closed connected subsets of C. Suppose that this sequence converges to a subset $A \subset C$ in the Hausdorff metric. Then A is connected.*

Proof: If A is disconnected, there are disjoint open sets $U, V \subset C$ such that $A \subset U \cup V$, and $A \cap U$ and $A \cap V$ are both not empty. The following pairs of sets are compact and disjoint:

$$(A, C - U - V) \qquad (A \cap U, C - U), \qquad (A \cap V, C - V).$$

(The set $A \cap U$ is compact because $A \cap U = A - V$. Similarly for $A \cap V$.) Hence, there is some $\epsilon > 0$ such that every point in the first set of a pair is at least ϵ from every point in the second pair. Therefore, for all sufficiently large n, the set A_n intersects both U and V. Since A_n is connected, this is only possible if $C - U - V$ contains a point $x_n \in A_n$. But then x_n is at least ϵ from A, independent of the choice of n. This contradicts the fact that $A_n \to A$ in the Hausdorff metric. ♠

Remarks:
(1) Since C is compact, the set A must be non-empty, by the Bolzano-Weierstrass Theorem.
(2) In our application the sets A_n will be path connected. However, there is no guarantee that the limit A is path connected as well. Consider a sequence of path approximations to the topologist's sine curve.

2.2 The Circular Invariant

Let S^1 be the unit circle. Let $|\alpha|$ denote the arc length of an arc $\alpha \subset S^1$. Let $\Sigma \subset (S^1)^4$ denote the subset of distinct labeled quadruples, which go counterclockwise around S^1. We call these *cyclic quadrilaterals*. Let σ_k be the kth vertex of σ. Any $\sigma \in \Sigma$ defines arcs $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ with $\alpha_k \subset S^1 - \sigma$ having endpoints σ_k and σ_{k+1}. The indices are taken mod 4. We write $\alpha_k(\sigma)$ when we want to emphasize the dependence on σ.

We define the *circular invariant* $\Lambda : \Sigma \to (0, \infty)$ by

$$\Lambda(\sigma) = \frac{|\alpha_0| + |\alpha_2|}{|\alpha_1| + |\alpha_3|}. \tag{2}$$

When σ consists of the vertices of a rectangle, $\Lambda(\sigma)$ is the aspect ratio of this rectangle. Otherwise $\Lambda(\sigma)$ is only vaguely related to an aspect ratio.

Lemma 2.2 *Let $\{\sigma_n\}$ be a sequence of cyclic quadrilaterals having diameter greater than some positive δ for all n, and circular invariant converging to 0. Then the arcs $\alpha_0(\sigma_n)$ and $\alpha_2(\sigma_n)$ shrink to points and the arcs $\alpha_1(\sigma_n)$ and $\alpha_3(\sigma_n)$ remain uniformly large.*

Proof: The hypotheses imply that $\lim_{n\to\infty} |\alpha_0(\sigma_n)| + |\alpha_2(\sigma_n)| = 0$. By the triangle inequality, $\min(|\alpha_1(\sigma_n)|, |\alpha_3(\sigma_n)|) > \delta/2$ for n large. ♠

We call $A \subset \Sigma$ *extensive* if A is connected and $\Lambda(A) = (0, \infty)$.

Lemma 2.3 *If A is extensive then all but at most 4 points of S^1 are vertices of members of A. If, additionally, A contains elements of arbitrarily small diameter then all but at most 2 points of S^1 are vertices of members of A.*

Proof: Let $\pi_k : A \to S^1$ be the map such that $\pi_k(\sigma) = \sigma_k$, the kth vertex. The set $J_k = \pi_k(\Sigma)$ is connected, and therefore either an open arc, a closed arc, a half-open arc, or all of S^1. If $B = S^1 - \bigcup J_k$ contains more than 4 points, there is some interval $B' \subset B$ which has an endpoint in common with J_k and an endpoint in common with J_{k+1} for some k. But then $B' \subset \alpha_k(\sigma)$ for all $\sigma \in A$. This bounds $\Lambda(A)$ away from 0 or ∞, depending on the parity of k, a contradiction.

If B has at least 3 points, then each member of A is a cyclic quadrilateral which nontrivially intersects each of 3 disjoint circular arcs. There is a uniform positive lower bound to the diameter of such cyclic quadrilaterals. ♠

2.3 A Compactness Result

For $K \geq 1$ let

$$\Sigma(K) = \Sigma \cap \Lambda^{-1}([1/K, K]). \tag{3}$$

Note that $\Sigma(K)$ is not compact because for any unit complex number u with positive imaginary part, the cyclic quadrilateral $(1, u, -1, \overline{u})$ lies in $\Sigma(1)$. In spite of this problem, we will prove a compactness result that involves $\Sigma(K)$.

Let γ be a general Jordan loop. We fix, once and for all, some homeomorphism $\phi : S^1 \to \gamma$. In [**Tv**], Tverberg gives a way to approximate γ by a sequence $\{\gamma_n\}$ of parametrized embedded polygons so that the parametrizations $\phi_n : S^1 \to \gamma_n$ converge uniformly to ϕ. Let Γ denote the set of $\sigma \in \Sigma$ such that $\phi(\sigma)$ is the vertex set of a rectangle in $G(\gamma)$, the space of rectangles gracefully inscribed in γ. Likewise define Γ_n relative to ϕ_n and γ_n.

Lemma 2.4 *For each fixed $K \geq 1$ there is a compact subset $C(K) \subset \Sigma$ such that $\Gamma_n \cap \Sigma(K) \subset C(K)$ for all n.*

Proof: If this is false then we can pass to a subsequence and cyclically relabel so that one of the following two things is true.

1. There is a sequence $\{\sigma_n\}$, with $\sigma_n \in \Gamma_n \cap \Sigma(K)$, such that $|\alpha_0(\sigma_n)| \to 0$.
2. There is a sequence $\{\sigma'_n\}$ with $\sigma'_n \in \Gamma_1 \cap \Sigma(K)$, such that $|\alpha_0(\sigma'_n)| \to 0$.

Consider Case 1. Let R_n be the rectangle in $G(\gamma_n)$ corresponding to σ_n We claim that $|\alpha_1(\sigma_n)| \to 0$ as well. If not, then the denominator of the expression for $A(\sigma_n)$ in Equation 2 is uniformly bounded away from 0. But then so is the numerator. Hence $|\alpha_2(\sigma_n)|$ is uniformly large. Since $\phi_n \to \phi$ uniformly, the side of R_n corresponding to $\alpha_0(\sigma_n)$ shrinks to a point but the opposite side corresponding to $\alpha_2(\sigma_n)$ does not. This is impossible for a sequence of rectangles. This proves our claim. But now we can repeat the same argument twice more to show that $|\alpha_k(\sigma_n)| \to 0$ for $k = 0, 1, 2, 3$. This contradicts the fact $\sum_{k=0}^{3} |\alpha_k(\sigma_n)| = 2\pi$.

Case 2 is really just an instance of Case 1 relative to the sequence of polygons $\{\gamma'_n\}$, the sequence of maps $\{\phi'_n\}$, and the limit $\phi' : S^1 \to \gamma'$. Here (somewhat trivially)

$$\gamma' = \gamma'_1 = \gamma'_2 = \gamma'_3 = ... = \gamma_1, \qquad \phi' = \phi'_1 = \phi'_2 = \phi'_3 = ... = \phi_1.$$

So, the argument in Case 1 takes care of Case 2. ♠

2.4 Limits of Hyperbolic Components

We keep the notation from the previous sections. In this section we will consider the special case that $G(\gamma_n)$ has a hyperbolic component H_n for all n. Recall that $A \subset \Sigma$ is extensive if A is connected and $\Lambda(A) = (0, \infty)$.

Lemma 2.5 *The subset $A_n \subset \Gamma_n$ corresponding to H_n is extensive.*

Proof: Each end of H_n consists of rectangles which accumulate on some chord of γ_n. Hence, the circular invariants of the corresponding cyclic quadrilaterals tend to 0 or ∞, with one case happening at one end and the other case happening at the other end. Hence $\Lambda(A_n) = (0, \infty)$. Also, A_n is homeomorphic to the arc H_n and hence also connected. ♠

Lemma 2.6 Γ *contains an extensive subset A.*

Proof: We keep the same notation as in the previous lemma. To make our proof more flexible, we only use the property that A_n is path connected, and contains a member with circular invariant n and a member with circular invariant $1/n$. Since A_n is connected, A_n contains some σ_n with circular invariant 1. For each $K = 1, ..., n$ we define $A_n(K)$ to be the minimal arc of A_n which contains σ_n and has endpoints with circular invariant $1/K$ and K respectively. By construction $A_n(K)$ is defined for $n \geq K$ and furthermore $A_n(K) \subset C(K)$, the compact set from Lemma 2.4. Finally,

$$A_n(1) \subset ... \subset A_n(n) \tag{4}$$

For fixed K, the sequence $\{A_n(K)\}$ is a sequence of closed connected subsets of the compact set $C(K)$. Using Cantor's diagonal trick, and compactness, we can find a subsequence so that for each $K = 1, 2, 3, ...$ the sequence $\{A_n(K)\}$ converges to some $A(K) \subset C(K)$ as $n \to \infty$. By Lemma 2.1 the set $A(K)$ is connected. Moreover, $A(K)$ contains elements of circular invariant K and $1/K$. Each $\sigma \in A(K)$ is such that $\phi(\sigma)$ is the vertex set of a non-degenerate limit of gracefully inscribed rectangles. Hence $\sigma \in \Gamma$. In short, $A(K) \subset \Gamma$. Equation 4 gives us $A(2) \subset A(3) \subset A(4)....$ The nested union of connected sets is connected. Therefore $A = \bigcup_K A(K)$ is connected. By construction A is extensive and $A \subset \Gamma$. ♠

2.5 Limits of Elliptic Components

We continue with the notation above. This time we treat the special case where $G(\gamma)$ has an elliptic component E_n for all n. We call the sequence $\{E_n\}$ *steady* if there is a uniform positive lower bound to the side length of any rectangle in any $G(\gamma_n)$, independent of n, and otherwise *wobbly*.

Lemma 2.7 *If $\{E_n\}$ is wobbly then Γ contains an extensive set A.*

Proof: Let $A_n \subset \Sigma$ be the set which corresponds to E_n. Again, A_n is connected. It cannot be the case that the circular invariants of members of A_n are uniformly bounded away from 0 and ∞. Otherwise, the lack of diameter bound contradicts Lemma 2.4. Therefore, after taking a subsequence, we can arrange that A_n either has a member with circular invariant n or a member with circular invariant $1/n$. Given the invariance of A_n under cyclic relabeling, we see that A_n has a member with circular invariant $1/n$ and a member with circular invariant n. This is all we used in the proof of Lemma 2.6. So, the same proof works here as well. ♠

Lemma 2.8 *We have $1 \in \rho(E_n)$ and $V(E_n, \gamma_n) = \gamma_n$.*

Proof: Recall that ρ is the aspect ratio. Since E_n is invariant under cyclic relabeling, we have that $r \in \rho(E_n)$ iff $1/r \in \rho(E_n)$. Since E_n is connected, $1 \in \rho(E_n)$. This is the first claim.

Let $v_k(R)$ denote the kth vertex of a rectangle R. We take indices mod 4. Choose any rectangle $R_0 \in E_n$. Since E_n is an elliptic component, there is a path $\{R_t |\ t \in [0,1]\}$ of rectangles in E_n such that $v_k(R_t)$ connects $v_k(R_0)$ and $v_{k+1}(R_0)$. We write $v_k(t) = v_k(R_t)$. It suffices to prove $\gamma_n = \bigcup v_k([0,1])$.

For the proof, we identify γ_n with $\boldsymbol{R}/4\boldsymbol{Z}$ so that the vertices of R_0 are $[0],[1],[2],[3]$. The path v_k connects $[k]$ to $[k+1]$. Let $\widehat{v}_k : [0,1] \to \boldsymbol{R}^2$ be the lift of v_k such that $\widehat{v}_k(0) = k$. Note that $\widehat{v}_k(0) \neq \widehat{v}_k(1) \in \boldsymbol{Z}$. Hence the interval $I_k = [\widehat{v}_k(0), \widehat{v}_k(1)]$ has length at least 1. We have

$$\widehat{v}_0(t) < \widehat{v}_1(t) < \widehat{v}_2(t) < \widehat{v}_3(t) < \widehat{v}_1(t) + 4. \quad \forall t \in [0,1]. \tag{5}$$

Equation 5 holds for $t = 0$ and any failure at time t would result in the points $\{v_k(t)\}$ not being distinct. Hence $I_{k+1} = I_k + 1$. This immediately implies that $\bigcup I_k$ contains an interval of length 4 and hence so does $\bigcup \widehat{v}_k([0,1])$. Hence $\boldsymbol{R}/4\boldsymbol{Z} \subset \bigcup v_k([0,1])$. ♠

Lemma 2.9 *If $\{E_n\}$ is steady then $G(\gamma)$ contains a compact connected subset S such that $1 \in \rho(S)$ and $V(\gamma, S) = \gamma$.*

Proof: Let $A_n \subset \Sigma$ be the subset corresponding to E_n. By hypotheses there is a single compact subset $C \subset \Sigma$ such that $A_n \subset C$ for all n. Passing to a subsequence, we take the Hausdorff limit $A = \lim A_n$. The set A is connected, by Lemma 2.1. We let $S = \phi(A)$. For the same reason as in the hyperbolic case, $S \subset G(\gamma)$.

Since $1 \in \rho(E_n)$ for all n, we can find some square in S corresponding to a limit of uniformly large squares in E_n. Hence $1 \in \rho(S)$.

Let $p \in \gamma$ be any point. Let $p_n \in \gamma_n$ be such that $p_n \to p$. Since $V(E_n, \gamma_n) = \gamma_n$ we can find a rectangle $R_n \in E_n$ such that p_n is a vertex of R_n. There is a uniform lower bound to the side lengths of these rectangles. Hence, any limit $\lim R_n$ will be a rectangle in S having p as a vertex. Hence $V(S, \gamma) = \gamma$. ♠

2.6 The Main Argument

Perturbing our polygons if necessary, we can assume that each γ_n satisfies Theorem 1.7. Passing to a subsequence, we reduce to either the hyperbolic case considered above, the steady elliptic case, or the wobbly elliptic case.

Case 1: In the steady elliptic case, Lemma 2.9 gives Option 1 of the Trichotomy Theorem.

Case 2: In the hyperbolic case or the wobbly elliptic case, Lemma 2.6 or Lemma 2.7 guarantees that Γ contains an extensive set A. Let $S \subset G(\gamma)$ be the subset corresponding to A. Suppose that there is a uniform positive lower bound to the diameters of members of A. We show that Option 2 of the Trichotomy Theorem holds.

By Lemma 2.3, at most 4 points of S^1 are not vertices of members of A. Hence $V(S, \gamma)$ contains all but at most 4 points of γ.

Since S is connected, $\rho(S)$ is connected. Since A is extensive, A contains a sequence $\{\sigma_n\}$ of circular quadrilaterals whose circular invariant tends to 0. By Lemma 2.2, this is only possible if the two arcs $\alpha_0(\sigma_n)$ and $\alpha_2(\sigma_n)$ in Equation 2 shrink to points and the other two arcs remain uniformly long. But then the aspect ratios of the corresponding rectangles tend to 0. Hence

$\rho(S)$ contains points arbitrarily near 0. The same argument shows that $\rho(S)$ contains points arbitrarily near ∞. Hence $\rho(S) = (0, \infty)$.

Case 3: The only remaining case is that Γ contains an extensive set A without the positive lower diameter bound. In this case, Lemma 2.3 shows that at most 2 points of S^1 are not vertices of members of A. Hence $V(S, \gamma)$ contains all but at most 2 points of γ. Since A contains members of every sufficiently small diameter, the set S does as well. This gives us Option 3 of the Trichotomy Lemma.

2.7 Non-Atomic Measures

Here we prove Corollary 1.4. Suppose that μ is a non-atomic probability measure on γ. Call a quadrilateral gracefully inscribed in γ *nice* if it cuts γ in such a way that opposite arcs have μ-measure 1/2. We are looking for a nice inscribed rectangle. Since μ is non-atomic, we can choose our homeomorphism ϕ so that ϕ pushes forward arc length on S^1 to $2\pi\mu$. Then nice rectangles correspond to elements of Γ having circular invariant 1.

In Cases 2 and 3 of our proof above, the sets A are extensive and have such cyclic quadrilaterals. So, Corollary 1.4 is true in Cases 2 and 3 above.

For Case 1, we revisit the proof of Lemma 2.9. Since A_n is invariant under cyclic relabeling, we have $1 \in \Lambda(A_n)$. So, by Lemma 2.4 we can take a limit and get $1 \in \Lambda(A)$. The corrsponding cyclic quadrilateral in A corresponds to a nice rectangle in the set S.

3 The Parity Equation

3.1 Outline of Proof

In this chapter we deduce Equation 1 from Theorem 1.5. Let $\mathcal{P}$ be the open dense set of polygons from Theorem 1.5. We fix some $\gamma \in \mathcal{P}$ for the entire argument. The space $I(\gamma)$ of labeled inscribed rectangles is a 1-manifold, by Theorem 1.5. The cyclic group $\boldsymbol{Z}/4$ acts on $I(\gamma)$ by cyclically relabeling the rectangles. Again, the labeling of a rectangle goes counterclockwise around the the rectangle. This is a free action: No point of $I(\gamma)$ is fixed by the relabeling.

For emphasis, we call the components of $I(\gamma)$ *labeled*. An *unlabeled* component is the orbit of a labeled component under the labeling action. We define the *order* of a labeled component to be the number of labeled components in its orbit – either 1, 2, or 4.

We say that a labeled rectangle R is *associated* with the labeled component that contains the point representing R. We say that a labeled rectangle R is associated to an orbit of a labeled component if it is associated to one of the labeled components in the orbit. Finally, we say that an unlabeled rectangle is associated to an unlabeled component if the corresponding labeled rectangles are associated with the corresponding orbit.

Below we will prove the following 4 claims.

1. The number of unlabeled inscribed squares associated to an unlabeled hyperbolic component is odd.

2. The number of unlabeled inscribed squares associated to any other unlabeled arc component is even.

3. The number of unlabeled inscribed squares associated to an unlabeled elliptic component is odd.

4. The number of unlabeled inscribed squares associated to any other unlabeled loop component is even.

Combining these claims, we see that the total number of unlabeled inscribed squares, namely Ω, has the same parity as $\Omega_H + \Omega_E$, the total number of unlabeled hyperbolic components plus the total number of unlabeled elliptic components. This is exactly Equation 1.

3.2 The Arc Components

Lemma 3.1 *Every arc component of $I(\gamma)$ has order 4.*

Proof: Recall that $I(\gamma)$ is naturally a subset of $\boldsymbol{R}^8$. Given an arc component ζ, there are two points ζ_1, ζ_2 in $\boldsymbol{R}^8$ corresponding to the ends of ζ. Up to cyclic relabeling, each of these points has the form (a, b, a, b, c, d, c, d) where $(a, b) \neq (c, d)$. Each of these points encodes the chord of γ corresponding to an end of ζ, and $\zeta_1 \neq \zeta_2$.

Suppose that $\psi(\zeta) = \zeta$ for some cyclic relabeling map ψ. Given the form of our points, we see that that $\zeta_j \neq \psi(\zeta_j)$. This means that ψ must interchange ζ_1 and ζ_2. But this means that ψ is a homeomorphism of the arc ζ which swaps its ends. This situation forces ψ to fix a point of ζ. This is impossible. ♠

Lemma 3.2 *Each labeled hyperbolic component contains an odd number of labeled inscribed squares and any other labeled arc component contains an even number of labeled inscribed squares.*

Proof: Let ζ be a labeled hyperbolic arc component. Let $\rho : \zeta \to (0, \infty)$ be the aspect ratio function. By definition $\rho(p) = 1$ if and only if p represents a square. At one end of ζ, the value of ρ is less than 1. At the other end, the value of ρ is greater than 1. Given that ρ is injective in a neighborhood of each point of $\rho^{-1}(1)$, this means that $\rho = 1$ an odd number of times on ζ. The argument for the non-hyperbolic arcs is the same except that ρ is either greater than 1 at both ends of ζ or less than 1 at both ends. ♠

Proof of Claim 1: Let $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ be the hyperbolic components comprising an orbit. There is some odd k such that ζ_1 has k labeled inscribed squares associated to it. But then, by symmetry, the same holds for the other components. Hence, there are a total of $4k$ labeled inscribed squares associated to these components. But this means that there are k unlabeled inscribed squares associated to these components. ♠

Proof of Claim 2: The proof of Claim 2 is the same as the proof of Claim 1 except that now k is even. ♠

3.3 The Loop Components

Lemma 3.3 *A labeled elliptic component contains $4k$ labeled inscribed squares for some odd integer k.*

Proof: Let ζ be some labeled elliptic component and r_0 be the point of ζ that has aspect ratio less than 1. Let r_1, r_2, r_3 be the successive images of r_0 under the relabeling map. Let ζ_k be the arc of ζ bounded by r_k and r_{k+1} with indices taken mod 4. Consider the restriction of the aspect ratio function ρ to ζ_0. We have $\rho(r_1) = 1/\rho(r_0)$. So, as we trace out ζ_0 from r_0 to r_1 we see that ρ starts out less than 1 and ends up greater than 1. Hence ρ attains the value 1 an odd number k of times on ζ_1. By symmetry, ρ attains the value 1 exactly k times on each arc ζ_k. This give a total of $4k$. ♠

Proof of Claim 3: Let ζ be some labeled elliptic component. The orbit of ζ is just ζ itself. We have just seen that the number of labeled inscribed squares associated to ζ is $4k$ for some odd k. But then the number of unlabeled inscribed squares associated to ζ is k. ♠

Lemma 3.4 *A labeled loop component contains $2k$ labeled inscribed squares for some integer k. If the component has order 2 then k is even.*

Proof: Let ζ be such a component. Since ζ is a topological loop and ρ is injective in a neighborhood of each point of $\rho^{-1}(1)$, the map ρ attains the value 1 an even number of times. This is the first statement.

Suppose then that ζ has order 2. Let r_0 be some point of ζ such that $\rho(r_0) \neq 1$. Since ζ has order 1, the element of $\boldsymbol{Z}/4$ which sents vertex 0 to vertex 2 must be the one which stabilizes ζ. Let r_1 be the image of r_0 under this relabeling element. Both r_0 and r_1 have the same aspect ratio. Hence ρ attains the value 1 an even number of times on each of the arcs of ζ joining r_0 to r_1. ♠

Proof of Claim 4: Let ζ be a labeled loop component which is not elliptic. Regardless of whether ζ has order 1 or 2, the preceding lemma says that there are $8h$ labeled squares associated to the orbit of ζ. Hence there are an even number of unlabeled squares associated to the orbit of ζ. ♠

3.4 Discussion

The analysis above combines with Theorem 1.6 to prove the follownig result.

Corollary 3.5 *For γ in $\mathcal{P}$ the space $G(\gamma)$ contains an odd number of squares.*

Proof: The analysis above shows that each non-global component of $I(\gamma)$ contributes an even number to the total number of unlabeled inscribed squares. By Theorem 1.6, therefore, $I(\gamma) - G(\gamma)$ contains an even number of inscribed squares. Hence $G(\gamma)$ contains an odd number of inscribed squares. ♠

Here is a way to deduce Theorem 1.7 from Corollary 3.5 without appealing to Theorem 1.6. The same argument as above establishes Equation 1 when we make the counts with respect to components in $G(\gamma)$. So, if we already know that there are an odd number of gracefully inscribed squares, then our new count implies $G(\gamma)$ has either a hyperbolic component or an elliptic component. This approach to Theorem 1.7 is more direct, and we originally took it. However, our direct proof that $G(\gamma)$ has an odd number of squares, a homotopy argument, was rather tedious.

4 The Moduli Space

4.1 Inscribing Rectangles in Four Lines

For this chapter we work in the complex plane $\boldsymbol{C}$. Let $L = (L_0, L_1, L_2, L_3)$ be a quadruple of lines in $\boldsymbol{C}$. We assume throughout the chapter that these lines are in general position. We say that a rectangle R *is gracefully inscribed in* L if the vertices (R_0, R_1, R_2, R_3) go cyclically around R (either clockwise or counterclockwise) and satisfy $R_i \in L_i$ for $i = 0, 1, 2, 3$. We let $G(L)$ denote the set of rectangles gracefully inscribed in L. We think of $G(L)$ as a subset of $\boldsymbol{R}^8$. In [**S**] I worked out quite a bit of the structure of $G(L)$. Here I will give a more abstract and less detailed treatment, but when relevant I will point out the stronger results that appear in [**S**].

We define the *aspect ratio* $\rho : G(L) \to \boldsymbol{R}$ by the formula

$$\rho(R) = \pm \frac{|R_2 - R_1|}{|R_1 - R_0|}. \tag{6}$$

The sign is -1 if R is clockwise ordered and $+1$ if R is counterclockwise ordered. We allow ρ to be both positive and negative, though ultimately we just care about the case $\rho > 0$.

Lemma 4.1 *Let $G(L, \rho)$ denote the subset of $G(L)$ consisting of rectangles having aspect ratio ρ. For generic choice of L, the space $G(L, \rho)$ has at most one element for every choice of ρ.*

Proof: Let R be a rectangle in $G(L)$. We have

$$R_2 - R_1 = i\rho(R_1 - R_0), \qquad R_3 - R_2 = R_1 - R_0.$$

Writing this as a matrix equaion, we have $(R_2, R_3) = M(R_0, R_1)$, where

$$M = \begin{bmatrix} -i\rho & 1 + i\rho \\ 1 - i\rho & i\rho \end{bmatrix}. \tag{7}$$

Note that $\det(M) = 1$ and $\text{trace}(M) = 0$. This means that M is always invertible and indeed an involution. Let $\Pi_{12} = L_0 \times L_1$ and $\Pi_{34} = L_2 \times L_3$. These are both totally real planes in $\boldsymbol{C}^2$. The solutions we seek are the points of $M(\Pi_{12}) \cap \Pi_{34}$. These two planes are either disjoint, or intersect in a d-dimensional affine subspace for $d = 0, 1, 2$.

The case $d = 2$ is certainly not generic, and we rule out the case $d = 1$ with a dimension count. The space of rectangles of aspect ratio ρ is 5-dimensional. So, the space of pairs of rectangles having aspect ρ is $(9 = 5 + 5 - 1)$-dimensional. Given two such rectangles R and R', we can recover the quadruple L by letting L_k be the line through R_k and R'_k. This accounts for all quadruples of interest. The space of rectangles of aspect ratio ρ gracefully inscribed in L is 1-dimensional affine subspace. Thus, we have overcounted the quadruples of lines of interest by 2 dimensions: Any pair of rectangles in the family would produce the same quadruple. This the space of quadruples containing infintely many rectangles of the same aspect ratio has dimension 7. On the other hand, the space of quadruples of lines has dimension 8. ♠

Remark: In [**S**] I show that $G(L)$ contains infinitely many rectangles of the same aspect ratio if and only if the line through $L_0 \cap L_1$ and $L_2 \cap L_3$ is perpendicular to the line through $L_1 \cap L_2$ and $L_3 \cap L_0$.

Since L is generic, $G(L)$ contains exactly one rectangle R_ρ of aspect ratio ρ provided that it contains any. Let $\rho(L)$ denote the set of aspect ratios of rectangles in $G(L)$. We define the point $\phi_k(\rho) \in L_k$ denote the kth vertex of R_ρ. This gives us a map $\phi_k : \rho(L) \to L_k$. We call ϕ_k a *vertex map*.

Let $\mathcal{L}$ be the space of generic quadruples. We can identity this space with an open subset of $\boldsymbol{R}^8$. We can think of our vertex map ϕ_k as a map $\mathcal{L} \times \boldsymbol{R} \to \boldsymbol{C}$. The point $\phi_k(L, \rho)$ is the vertex of the rectangle R_ρ defined relative to the configuration L. The domain for ρ_k is naturally the fiber bundle $\mathcal{E}$ over $\mathcal{L}$ whose fiber is $\rho(L)$.

Lemma 4.2 *The space $\mathcal{E}$ is an open subset of $\boldsymbol{R}^9$ and each ρ_k is an analytic function on $\mathcal{E}$.*

Proof: Referring to Lemma 4.1, the existence of R_ρ means that the two planes $M(\Pi_{12})$ and Π_{34} are transverse. A small change in ρ or in L does not change that fact. Hence $\mathcal{E}$ is open in $\boldsymbol{R}^9$. The desired point can be found using linear algebra with inputs that vary analytically with the coordinates on $\mathcal{E}$. Everything in sight is algebraic, and hence analytic. ♠

Remark: In [**S**] I show that the set of centers of rectangles in $G(L)$ is a hyperbola minus 2 points, namely those corresponding to degenerate rectangles

of aspect ratio 0 and ∞. There are two unequal values a_1 and a_2, corresponding to the points at infinity of the hyperbola, such that $\rho(L) = \boldsymbol{R} - \{0, a_1, a_2\}$. Moreover $a_1 a_2$ equals the cross ratio of the slopes of the lines of L.

There is one degenerate case we need to consider. We say that a *repeating quadruple* is one of the form

$$(L_0, L_0, L_1, L_2), \quad (L_0, L_1, L_1, L_2), \quad (L_0, L_1, L_2, L_2), \quad (L_0, L_1, L_2, L_0). \tag{8}$$

Here L_0, L_1, L_2 are distinct and non-parallel lines. We call these kinds of quadruples *repeating quadruples*. We make the same definition for $G(L)$. The same results as above apply in this case. Indeed, it never happens that $G(L)$ contains infinitely many rectangles of the same aspect ratio.

Permutation Trick: We have now defined two kinds of quadruples, the generic ones and the generic repeating ones. So far we have been talking about gracefully inscribed rectangles. Since we are interested in all inscribed rectangles, rather than just the gracefully inscribed ones, we note that the permutation of a generic quadruple is still generic and the cyclic or dihedral permutation of a repeating quadruple is still a repeating quadruple. For instance, if we are interested in the space $G'(L)$ of rectangles having the property that $R_0 \in L_1$ and $R_1 \in L_3$ and $R_2 \in L_0$ and $R_3 \in L_2$ then we are really considering the space $G(L')$, where $L' = (L_1, L_3, L_0, L_2)$ is the suitable permutation of the lines of L. The space $I(\gamma)$ divides into different subspaces, depending on the combinatorics of the labelings. The space $G(\gamma)$ is one of these. When proving things for $I(\gamma)$, we will often specialize to the case of $G(\gamma)$, with the understanding that the permutation trick just discussed promotes the proof we give for $G(\gamma)$ to a proof for $I(\gamma)$.

Regular and Singular Values: Since ϕ_k is analytic, there are finitely many values $b_1, ..., b_\ell \in \rho(L)$ such that $d\phi_k/da = 0$. Here we are differentiating with respect to the aspect ratio parameter. We call $b_1, ..., b_\ell$ the *singular ratios* for ϕ_k and we call their images on L_k the *singular images*. We call $a \in \boldsymbol{R}$ a *regular ratio* if it is not a singular ratio. Geometrically, the kth vertex of R_a varies monotonically on L_k near a regular value a. We call the image of a regular ratio a *regular image*.

4.2 The Polygon Space

In this section, we define our space $\mathcal{P}$ of polygons and then show that it is open and dense in the space of all polygons. What we really mean is that there is a space $\mathcal{P}(N)$ for each $N \geq 3$, which is a subset of the finite dimensional space of all labeled N-gons, and $\mathcal{P}(N)$ is open and dense in this space. The space of all labeled N-gons is simply $\boldsymbol{R}^{2N}$. So, our space $\mathcal{P}(N)$ is an open dense subset of $\boldsymbol{R}^{2N}$.

Let γ be a polygon, with sides $E_1, ..., E_N$. We insist first of all that no two sides of γ are parallel. This assumption implies that any rectangle gracefully inscribed in γ has its vertices in at least 3 distinct edges. We say that an *associated quad* $L = (L_0, L_1, L_2, L_3)$ is a quadruple of lines, cyclically ordered (counterclockwise) and extending some sides of γ, which is either repeating or ordinary in the sense above. We say that γ belongs to $\mathcal{P}$ if

1. All associated quads are generic or repeating.
2. No inscribed rectangle has more than 1 vertex in common with γ.
3. No inscribed square has a vertex in common with γ.
4. No vertex of γ is a singular image with respect to an associated quad.

Lemma 4.3 *$\mathcal{P}$ is open and dense in the space of all polygons.*

Proof: Condition 1 clearly holds on an open dense set. The remaining conditions are open because their negation is closed. For instance, if we have a convergent sequence of N-gons having an inscribed square that shares a vertex with the polygons, then we can take a limit and get such a square on the limiting polygon. The other conditions are similar.

We will deal with Condition 2. Given any pair v_1, v_2 of vertices of γ, let e be the edge joining v_1 and v_2 and let λ_1 and λ_2 be the lines perpendicular to e through v_1 and v_2 respectively. Let λ_3 be the circle having e as a diameter. We can perturb so that $\lambda_i \cap \gamma$ is always a finite set of points. We can further perturb so that there are no points $v_i \in \lambda_i \cap \gamma$ for $i = 1, 2$ or $v_1, v_2 \in \gamma_3$ such that $\|v_1 - e_1\| = \|v_2 - e_2\|$. Thus, after finitely many steps, we get Condition 2 and keep Condition 1.

Conditions 3 and 4 involve a single vertex at a time. To show density for these, we introduce a *slide move*. This is defined relative to a vertex v and one of the edges e incident to v. We replace v by a vertex $v' \in e$ very close

to v and then consider the new polygon having v' as a vertex in place of v and all other vertices the same.

If we have a polygon which fails to have one of the conditions above, we can associate that failure to a triple (v, e, L) where v is an involved vertex, e is an involved edge, and L is an associated quadruple one of whose sides extends e. In case the same problem – e.g. a square sharing a vertex with the polygon – involves more than one triple (v, e', L') we count this as a separate problem. Each slide move, if done with respect to a sufficiently nearby vertex, removes one of the problems and does not create any new ones.

For instance, given (v, e, L), there exist points $v' \in e$ arbitrarily close to v such such that the square in $G(L)$ does not contain v' and v' is a regular value for the relevant vertex map. This follows from the analyticity of everything in sight: The problem points on e are isolated.

So, we go around making small perturbations fixing one problem at a time until we are done, and we can make these perturbations as small as we like. ♠

4.3 The Manifold Structure

Now we prove Theorem 1.5. Let γ be a polygon in $\mathcal{P}$.

Lemma 4.4 *The space $I(\gamma)$ is a piecewise smooth manifold.*

Proof: We will prove this for $G(\gamma)$. As discussed above, the permutation trick promotes the proof to a proof for $I(\gamma)$. We have a partition

$$G(\gamma) = G_0(\gamma) \cup G_1(\gamma). \tag{9}$$

The points of $G_k(\gamma)$ correspond to gracefully inscribed rectangles which have k points in common with the vertex set of γ.

Each point of $G_0(\gamma)$ corresponds to a rectangle of $G(L)$ for some unique associated quadruple of lines. All nearby points of $G_0(\gamma)$ are associated to the same quadruple of lines. Thus $G_0(\gamma)$ is open in $G(\gamma)$, and every point of $G_0(\gamma)$ has a neighborhood which is just a copy of a neighborhood of the corresponding point of $G(L)$. So, by the results in §4.1, the set $G_0(L)$ is a smooth manifold and the aspect ratio function ρ gives a coordinate chart.

Let $p \in G_1(L)$. Let R be the associated rectangle and let $v \in R$ be the vertex of R which is also a vertex of γ. There are exactly 2 associated

quadruples L and L' such that the rectangle R associated to p lies in $G(L)$ and $G(L')$. After cyclically relabelling, we can arrange that L_0 and L_0' are the two lines extending the edges of γ incident to v, and $L_j' = L_j$ for $j = 1, 2, 3$.

Let U and U' denote small open subsets of p in $G(L)$ and $G(L')$ respectively. Each member of $G(\gamma)$ sufficiently close to p lies in one of $G(L)$ or $G(L')$, so a small neighborhood of p in $G(\gamma)$ is given by

$$(U \cap G(\gamma)) \cup (U' \cap G(\gamma)).$$

Because p is a regular image with respect to L or L', the set U intersects $G(\gamma)$ in a half-open interval having p as endpoint. The idea here is that as we vary the point in $G(L)$ the vertex near p move monotonically along L_0, spending half the time on the edge of γ contained in L_0 and half the on L_0 outside this edge. The same goes for U'. Thus, the two half-open arcs fit together to give a neighborhood of p in $G(\gamma)$ homeomorphic to an arc.

We have shown that every point of $G(\gamma)$ has an arc neighborhood, and every point of $G_0(\gamma)$ is smooth. Finally, it follows from the analyticity of the vertex maps that there are only finitely many rectangles of $G(\gamma)$ having any given point of γ as a vertex. In particular, $G_1(\gamma)$ is just a finite set of points. Hence $G(\gamma)$ is a piecewise smooth 1-manifold. ♠

Lemma 4.5 *The aspect ratio function ρ is locally injective at each smooth point of $I(\gamma)$, and $\rho^{-1}(1)$ consists entirely of smooth points.*

Proof: The set of smooth points is precisely the set $G_1(\gamma)$ considered above. The restriction of ρ to a small neighborhood of such a point coincides with the restriction of ρ to some neighborhood of the corresponding point of $I(L)$. This restriction is injective by Lemma 4.1. The second statement of the lemma is exactly Condition 3. ♠

Lemma 4.6 *Each arc component of $I(\gamma)$ is proper.*

Proof: We prove this for $G(\gamma)$ and then use the permutation trick to promote the proof to one for all of $G(\gamma)$. We first recall what we are trying to prove. We have $G(\gamma) \subset \boldsymbol{R}^8$. Let $A \subset G(\gamma)$ be an arc component. Let $\partial A = \overline{A} - A$ denote the boundary of A in $\boldsymbol{R}^8$. We will show that ∂A consists of 2 distinct points, both representing chords of γ.

In the proof of Lemma 4.4 we saw that the space $G_1(\gamma)$ is a finite set of points. So, if a rectangle in $G(\gamma)$ has sufficiently large or small aspect ratio it must be a smooth point. This means that there are two unique associated quadruples L and L' such that the ends of A respectively lie in $G(L)$ and $G(L')$. If the rectangles at one end of A accumulate to something other than a line segment, then we can take a subsequential limit of these uniformly large and fat rectangles to get another member of $G(L)$. This rectangle would also grace L, and hence γ, and it would have a neighborhood in $G(\gamma)$ that overlaps with A. This is a contradiction. Hence, as we exit an end of A the corresponding rectangles accumulate on a line segment.

For the end of A associated to L, the relevant rectangles accumulate to a chord that has one vertex on two consecutive lines of L and one vertex on the other two. This chord is uniquely determined by L and, by general position, uniquely determines L amonst all associated quads. All the same remarks apply to the other end of A, which is associated to the quadruple L'.

It remains to show that our chords are distinct. If not, then $L = L'$ and the two ends, which both have their elements in $G(L) = G(L')$, have some rectangles in commn. This is a contradiction. ♠

Remark: The ends of each proper arc of $G(\gamma)$ are critical points for the distance function $d : \gamma \times \gamma \to [0, \infty)$, at least after one makes a suitable definition for what this means in the polygonal case. After doing thousands of experiments, I noticed that one end of a proper arc is always a saddle point (i.e. a point with Morse index 1) and the other end is always either a maximum or a minimum (i.e. a point with Morse index 0 or 2.) I have no idea how to prove it, but this fact suggests hidden depths.

5 Global Components are Graceful

5.1 The Easy Part

In this chapter we prove Theorem 1.6. We say that a rectangle $R \in I(\gamma)$ is *ungracefully inscribed* γ if the cyclic order imparted on R from the ordering on γ is the clockwise ordering of the vertices of R. Let $G^*(\gamma)$ denote the space of rectangles which are ungracefully inscribed in γ.

Lemma 5.1 *A global component of $I(\gamma)$ lies in $G(\gamma)$ or $G^*(\gamma)$,*

Proof: Consider the hyperbolic case first. If the rectangles in the hyperbolic component A are neither gracefully nor ungracefully inscribed, then there are a pair of opposite sides of the rectangles such that the endpoints on one pair of opposite sides interlace on γ with the endpoints on the other pair. However, at one end of A, these edges are very short. This is only possible if the rectangles are shrinking to a single point. This is a contradiction.

When A is elliptic, we revisit the proof of Lemma 2.8. The same paths $\{v_k\}$ exist in the more general setting, but now there is some permutation $\{i_0, i_1, i_2, i_3\}$ of $\{0, 1, 2, 3\}$ such that $v_k(1) = [i_k]$. Equation 5 holds in this setting, and tells us that there are integers j_0, j_1, j_2, j_3 such that

$$i_0 + j_0 < i_1 + j_1 < i_2 + j_2 < i_3 + j_3 < i_0 + j_0 + 4.$$

Here $i_k + j_k = \widehat{v}_k(1)$. This forces $[i_0], [i_1], [i_2], [i_4]$ to be consecutive residue classes in $\boldsymbol{Z}/4$. But then (i_0, i_1, i_2, i_3) is a cyclic permutation of $(0, 1, 2, 3)$. This happens if and only if $A \in G(\gamma)$ or $A \in G^*(\gamma)$. ♠

Remark: The components $G(\gamma)$ and $G^*(\gamma)$ might look superficially similar, but actually they are quite different. For instance, $G^*(\gamma)$ is empty if γ is convex. Before we prove that $G^*(\gamma)$ has no global components, we explain why most of the Trichotomy Theorem follows from what we have already done. The definition of the circular invariant and the other proofs in §2, go through practically word for word if we work with components in $G^*(\gamma)$ rather than in $G(\gamma)$. Thus, if we use Lemma 5.1 in place of Theorem 1.6, we get the whole Trichotomy Theorem except that the last statment is weaker: $S \subset G(\gamma)$ or $S \subset G^*(\gamma)$.

5.2 The Elliptic Case

In this section we prove that $G^*(\gamma)$ contains no elliptic components. We will deduce this result from a theorem about inscribed triangles. We define gracefully and ungracefully inscribed triangles the same way as for rectangles. We say that an *essential graceful loop* (respectively *essential ungraceful loop*) is a continuous loop of gracefully (respectively ungracefully) inscribed triangles such that each vertex winds a nontrivial number of times around γ. We prove the following result.

Theorem 5.2 *No polygon has an ungraceful loop.*

If we had an elliptic component in $G^*(\gamma)$ we could look at the loop of triangles made from the first 3 points. The same lifting argument as in the proof of Lemma 5.1 shows that the kth vertex of the rectangle family winds around γ a nonzero number of times. So, we would get an essential ungraceful loop, contradicting Theorem 5.2. We prove Theorem 5.2 through two lemmas.

Lemma 5.3 *A polygon arbitrarily close to γ in the Hausdorff metric supports a graceful essential loop.*

Proof: We describe a motion of the points a, b, c. We can find an arbitrarily nearby polygon whose convex hull has 8 consecutive vertices $v_1, ...v_8$ which agree with the vertices of a regular polygon. We start with points a, b, c located at v_5, v_6, v_7. We then move c all the way around γ counterclockwise until $c = v_4$. Next, we move b around to v_3, then a around to v_2. Now we have a, b, c located at v_2, v_3, v_4. Finally, we slide this triangle over to its original location. ♠

If an elliptic component of $H(\gamma)$ lies in $G^*(\gamma)$ then γ supports an essential ungraceful loop. But then so do all polygons sufficiently near γ. (The new loops won't necessarily be comprised of right triangles, but we don't care.) But then we could have an example of a triangle which supports both a graceful essential loop and an ungraceful essential loop. So, the following lemma finishes the proof of Theorem 5.2.

Lemma 5.4 *No polygon can support both an essential graceful loop and an essential ungraceful loop.*

Proof: Let Ω denote the subset of distinct triples of points in γ, with the order induced by the ordering on γ. Let $\Omega_+ \subset \Omega$ be the subset corresponding to triangles having positive signed area. Likewise define Ω_-. Our graceful and ungraceful essential loops respectively define essential loops β_+ and β_- in Ω_+ and Ω_-. We can replace β_+ and β_- by nearby polygonal loops.

There are nonzero integers $n_\pm$ such that $n_+\beta_+$ and $n_-\beta_-$ are homologous. But then we can find a piecewise linear surface-with-boundary that has $n_+\beta_+$ and $n_-\beta_-$ as boundary components. The common boundary of Δ_+ and Δ_- is piecewise algebraic, and so (after we perturb to put things in general position) the intersection $\Delta_+ \cap \Sigma$ consists of finitely many piecewise smooth loops. The union of these loops is homologous to $n_+\beta_+$, so some component β_0 is essential.

Say that a *stick* is a triple of collinear points of γ. The loop β_0 corrsponds to a continuous family of sticks having the property that each point of the stick winds a nonzero number of times around γ. But then there will be a moment when the middle point of the stick will be a vertex of the convex hull of γ. At this moment, the other two points of the stick cannot lie on γ, and we have a contradiction. ♠

5.3 The Hyperbolic Case

Now we prove that $G^*(\gamma)$ has no hyperbolic component. The argument is similar. This time let Ω denote the set of quadruples of distinct points in γ, with the order induced by the ordering on γ. The space Ω is homeomorphic to the product of a 3-ball and a circle. Let Ω_+ (respectively Ω_-) denote the set of quadruples defining embedded quadrilaterals, not necessarily convex, whose ordering goes counterclockwise (respectively clockwise) around their boundary. Unlike in the elliptic case, the two sets Ω_+ and Ω_- do not partition Ω. That does not bother us.

We distinguish 2 special subsets of $\partial\Omega$. Let $\partial_0\Omega$ be the set of embedded quadruples (a, a, b, b) and let $\partial_1\Omega$ be the set of quadruples (a, b, b, a). A hyperbolic component defines a curve joining $\partial_0\Omega$ to $\partial_1\Omega$. The interior of this curve lies in Ω_+ when the hyperbolic component is graceful and in Ω_- when the hyperbolic component is ungraceful.

Lemma 5.5 *A polygon arbitrarily close to γ in the Hausdorff metric supports a path in Ω_+ joining $\partial_0\Omega$ to $\partial_1\Omega$.*

Proof: We describe a motion of points a, b, c, d. Make the same modification to γ as in the proof of Lemma 5.3. (We really don't need all 8 points here.) Start out with $a = b = v_3$ and $c = d = v_6$. First move b and c respectively to v_4 and v_5. Now move d all the way around γ to v_2. At every stage of this construction we have a convex quadrilateral. Now we have a, b, c, d at v_2, v_3, v_4, v_5. Finally, move b and c back to v_2 and v_3 respectively. Now we have $b = c = v_2$ and $a = d = v_5$. ♠

Lemma 5.6 *There cannot be paths both in Ω_+ and Ω_- joining $\partial_0\Omega$ to $\partial_1\Omega$.*

Proof: The proof here is similar to the elliptic case. We first perturb so that both paths are polygonal. We then observe that both paths represent the generator of the relative homology group

$$H_1(\Omega, (\partial_0(\Omega) \cup \partial_1(\Omega)) = \boldsymbol{Z}.$$

(The space in question is homotopy equivalent relative the boundary to an annulus relative its boundary.) So, we can build a piecewise linear surface-with-boundary Σ whose boundary is made up of a path $\partial_0\Omega$, a path in $\partial_1\Omega$, and our 2 curves. After we put things in general position, the intersection $\Sigma \cap \partial\Omega_+$ is a union of loops and arcs in Σ, one of which joins $\partial_0\Omega$ to $\partial_1\Omega$. Call this path β_0.

The quadrilaterals in β have 4 distinct points but are not embedded. One possibility is that the points are all collinear and the other possibility is shown in Figure 3.

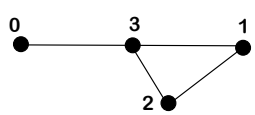

Figure 3: A degenerate quadrilateral

In case the points are not all collinear, there is always one segment that has another vertex between it. In Figure 3, the segment is v_0v_1 and the middle point is either v_2 or v_3. We call v_0, v_1 a *framing segment*. We can always cyclically relabel so that at some point along β_0 the segment v_0v_1 is a framing segment. Perturbing β slightly, we can arrange that the points along β corresponding to 4 collinear points are isolated. Such totally collinear configurations have codimension 1 in $\partial\Omega_+$.

Consider what happens to a configuration with v_0v_1 as a framing segment as we pass through a totally collinear configuration. Figure 4 shows the only 3 possibilities.

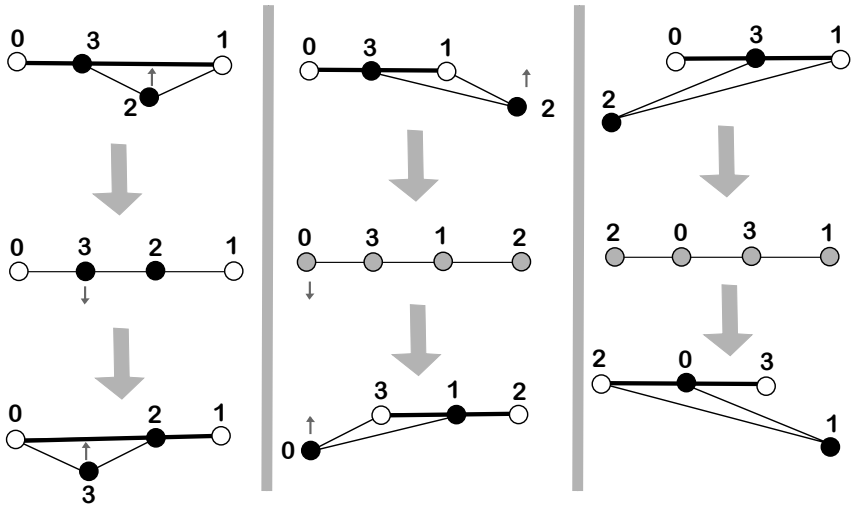

Figure 4: The only allowable transitions

The framing segment either remains v_0v_1 or else changes to v_2v_3. Therefore, one of the two segments v_0v_1 or v_2v_3 is the framing segment at each point of β_0. At $\beta_0 \cap \partial_0\Omega$, we have $v_0 = v_1$ and $v_2 = v_3$. But the distance between the point on the framing segment to either endpoint of the draming segment therefore tends to 0. This shows that there is a sequence of points in Ω converging to a point of $\partial_0\Omega$ such that the diameter of some 3 of the points tends to 0. This contradicts the fact that the points of $\partial_0\Omega$ come together in pairs and not in triples. ♠

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