CONTINUOUS FRAME DECOMPOSITION AND A VECTOR HUNT – MUCKENHOUPT – WHEEDEN THEOREM.

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1. Introduction

This paper deals with the weighted norm inequalities for Hilbert Transform with matrix-valued weights. The main problem can be formulated as follows. Let W be a $d\times d$ matrix weight, i.e. a L^1 function whose values are selfadjoint nonnegative $d\times d$ matrices. We suppose that the weight W is defined on the unit circle $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$.

Let $L^2=L^2(\mathbb{C}^d)$ be the space of square summable functions on $\mathbb{T}$ with values in $\mathbb{C}^d$, $H^2=H^2(\mathbb{C}^d)$ be the corresponding Hardy space of analytic functions and P_+ be the orthogonal projection in L^2 onto H^2. Let T denote the Hilbert Transform, $T=-iP_++i(I-P_+)$.

The question we are interested in is under what conditions on W the following weighted norm inequality for T holds (say for all $f\in L^2\cap L^\infty$):

$$\int_{\mathbb{T}}(W(\xi)Tf(\xi),Tf(t))dt\le C\int_{\mathbb{T}}(W(\xi)f(\xi),f(\xi))dm(\xi)$$

where m is the normalized ($m(\mathbb{T})=1$) Lebesgue measure on $\mathbb{T}$. Clearly this inequality is equivalent to the same inequality for P_+ (with another constant) .

If we define a weighted space $L^2(W)$ as the space of all measurable $\mathbb{C}^d$-valued functions on $\mathbb{T}$ satisfying

$$\|f\|^2_{L^2(W)}\stackrel{\text{def}}{=}\int_{\mathbb{T}}(W(\xi)f(\xi),f(\xi))dm(\xi)<\infty\,,$$

(of course we should factorize it over the subspace of functions of norm 0), then the last inequality means that T (or, equivalently P_+) is a bounded operator in $L^2(W)$.

If the dimension $d=1$ and everything is scalar-valued the answer is given by the famous Hunt — Muckenhoupt — Wheeden theorem, that says that the Muckenhoupt condition (A_2)

$$\sup_I\frac{1}{|I|}\int_I W\cdot\frac{1}{|I|}\quad\int_I W^{-1}<\infty$$

(supremum is taken over all intervals) is necessary and sufficient in this case.

In the matrix case it was conjectured by the first author in [3] and proved in our paper [4] that the vector Muckenhoupt condition $(\mathbf{A}_2)$,

$$\sup_I\left\|\left(\frac{1}{|I|}\int_I Wdm\right)^{1/2}\left(\frac{1}{|I|}\int_I W^{-1}dm\right)^{1/2}\right\|<\infty$$

is necessary and sufficient for the boundedness of Hilbert Transform in $L^2(W)$ with matrix weight. In this paper we present an alternative proof of that result.

The main technical tool we are using here is a matrix version of the Littlewood–Paley type inequality that gives an equivalent norm in the weighted L^2 space in terms of weighted L^2 norm of the derivative of the harmonic extension see Theorem 4.2 below. The scalar version was developed by us earlier in [5].

This equivalent norm inequality can be viewed as a continuous analogue of the wavelet type decomposition (Haar system is a Riesz basis in $L^2(W)$) that was used by us in [4]. But in this case the continuous "system of coefficients" (derivatives of harmonic extension) is over-determined, so it is more appropriate to call it continuous frame decomposition, see [6] about frames.

Although the main result about boundedness of the Riesz Projection is already known, we feel that the technique we use is of independent interest, and deserves separate consideration. In a sense the main result is Theorem 4.2 below about continuous frame decomposition, and boundedness of the Riesz Projection serves as an illustration of usefulness of this theorem.

2. Properties of matrix $(\mathbf{A}_2)$ weights.

Given a matrix weight W and a set $I \subset \mathbb{R}$ let us denote by $W(I)$ the integral $W(I) \stackrel{\text{def}}{=} \int_I W dm$ and by W_I the average value $W_I \stackrel{\text{def}}{=} W(I)/|I|$, where m and $|.|$ denotes normalized $(m(\mathbb{T}) = 1)$ Lebesgue measure on $\mathbb{T}$. Also for $\lambda \in \mathbb{D}$ let $W(\lambda)$ denote the harmonic (Poisson) extension of W at λ,

$$W(\lambda) := \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|1 - \overline{\lambda}\xi|^2} W(\xi) dm(xi)$$

It was shown in [4] that if the Riesz projection P_+, $P_+\left(\sum_{-\infty}^{\infty} \hat{f}(k) z^k\right) = \sum_0^{\infty} \hat{f}(k) z^k$ is bounded in the vector-valued weighted space $L^2(W)$ then the matrix weight W satisfies

$$\sup_{\lambda \in \mathbb{D}} \left\| W(\lambda)^{1/2} W^{-1}(\lambda)^{1/2} \right\| < \infty . \tag{2.1}$$

The last condition in turn implies

$$\sup_I \left\| \left[W_I\right]^{1/2} \left[(W^{-1})_I\right]^{1/2} \right\| , \tag{$\mathbf{A}_2$}$$

where the *supremum* is taken over all arcs $I \subset \mathbb{T}$. The last condition is called the vector Muckenhoupt condition $(\mathbf{A}_2)$, and the supremum is called the Muckenhoupt norm of W.

We will need some properties of Muckenhoupt weights. In particular that $(\mathbf{A}_2)$ implies (2.1). Of course it immediately follows from the fact that P_+ is bounded in the weighted space $L^2(W)$, but there exists a direct much simpler proof.

Lemma 2.1. *Let W be a nonnegative, measurable, $d \times d$ matrix function on a measure space $\mathcal{X}, \mu$. Then*

$$\int_{\mathcal{X}} \|W(t)\| d\mu(t) \le d \left\| \int_{\mathcal{X}} W(t) d\mu(t) \right\|$$

Proof. Since for a nonnegative $d \times d$ matrix A we have $\|A\| \le \operatorname{trace} A \le d\|A\|$, we can conclude that

$$\int_{\mathcal{X}} \|W(t)\| d\mu(t) \le \int_{\mathcal{X}} \operatorname{trace}(W(t)) d\mu(t) = \operatorname{trace}\left(\int_{\mathcal{X}} W(t) d\mu(t)\right) \le d \cdot \left\| \int_{\mathcal{X}} W(t) d\mu(t) \right\| .$$

□

Lemma 2.2. *Let W be a $d\times d$ matrix Muckenhoupt weight, and A be a positive nonsingular $d\times d$ (constant) matrix. Then the weight $\mathcal{W} := AWA$ also satisfies the Muckenhoupt condition ($\mathbf{A}_2$) with the same constant.*

Proof. Clearly, $\mathcal{W}_I = AW_IA$ and $(\mathcal{W}^{-1})_I = A^{-1}(W^{-1})_IA^{-1}$. Operator $B = (W_I)^{1/2}A$ is a Hermitian square root of $\mathcal{W}_I$, that means $B^*B = \mathcal{W}_I$. Therefore we can write a polar decomposition for the operator $B = (W_I)^{1/2}A = U(\mathcal{W}_I)^{1/2}$, where U is a unitary matrix. Similarly, for an operator $C = ((W^{-1})_I)^{1/2}A^{-1}$ one can write $C = V((\mathcal{W}^{-1})_I)^{1/2}$, where V again is a unitary matrix.

Therefore

$$BC^* = (W_I)^{1/2}\left((W^{-1})_I\right)^{1/2} = U(\mathcal{W}_I)^{1/2}\left((\mathcal{W}^{-1})_I\right)^{1/2}V^*,$$

and so

$$\left\|(W_I)^{1/2}\left((W^{-1})_I\right)^{1/2}\right\| = \left\|(\mathcal{W}_I)^{1/2}\left((\mathcal{W}^{-1})_I\right)^{1/2}\right\|.$$

□

The following 2 lemmas about scalar (A_2) weights are well known and their proofs are presented only to make the paper self-contained.

For an arc $I\subset\mathbb{T}$ let λ_I be the center point of the top cover of the corresponding "Carleson square", i.e. $1-|\lambda| = |I|$ and $\lambda/|\lambda|$ is the center of I.

Lemma 2.3. *Let a scalar weight w satisfies the Muckenhoupt (A_2) condition*

$$\sup_I\left\{w_I\cdot(w^{-1})_I\right\} =: C < \infty. \tag{A_2}$$

Then for any arc $I\subset\mathbb{T}$ the inequality $w(\lambda_I)\le K(C)w_I$ holds, where the constant K depends only on the Muckenhoupt norm C.

Proof. Given an ark I let kI, $k>0$ denote the arc with the same center of length $k|I|$ ($kI = \mathbb{T}$ for $k\ge 1/|I|$). It is easy to see from the form of Poisson kernel and the Muckenhoupt condition that

$$w(\lambda_I)\le A\cdot\sum_{k=0}^\infty 2^{-k}\cdot w_{2^kI}\le A\cdot\sum_{k=0}^\infty 2^{-k}\cdot\frac{w_{2^kI}}{(w^{-1})_{2^kI}}\cdot(w^{-1})_{2^kI}\le AC\cdot\sum_{k=0}^\infty 2^{-k}\Big((w^{-1})_{2^kI}\Big)^{-1}$$

The Muckenhoupt (A_2) condition implies that the weight w^{-1} is doubling, and therefore for any arc I we have $(w^{-1})_{2I}\ge(2-\varepsilon)^{-1}w_I$, where the constant ε depends only on the Muckenhoupt norm of w (or w^{-1}, what is equivalent). Therefore

$$w(\lambda_I)\le AC\cdot\sum_{k=0}^\infty 2^{-k}\Big((w^{-1})_{2^kI}\Big)^{-1}\le AC\cdot\sum_{k=0}^\infty 2^{-k}(2-\varepsilon)^k\Big((w^{-1})_I\Big)^{-1}\le C'(w^{-1})_I\le CC'w_I$$

□

Corollary 2.4. *If a $d\times d$ matrix weight W satisfies the vector Muckenhoupt condition ($\mathbf{A}_2$) then*

$$\sup_{\lambda\in\mathbb{D}}\|W(\lambda)^{1/2}W^{-1}(\lambda)^{1/2}\| < \infty \tag{$\mathbf{A}_2'$}$$

where the supremum depends on d and the Muckenhoupt norm of W.

We will call the condition $(\mathbf{A}_2')$ *conformally invariant* (or simply *invariant*) Muckenhoupt condition.

Proof. One can easily see from Lemma 2.1 in [4] (with $k = |I|^{-1/2}\chi_I$) that the Muckenhoupt condition is equivalent to the uniform boundedness of all averaging operators

$$f \mapsto \chi_I \frac{1}{|I|}\int_I f, \qquad f \in L^2(\mathbb{C}^d).$$

Fix any direction (unit vector) $\mathbf{e} \in \mathbb{C}^d$, and consider the restrictions of these operators on the subspace $\{f\mathbf{e} \ : \ f \in L^2\}$. It follows that for any $\mathbf{e} \in \mathbb{C}^d$, $\mathbf{e}\| = 1$ the scalar weight $w(t) = (W(t)\mathbf{e}, \mathbf{e})$ is a Muckenhoupt weight with the same or better constant as W.

Lemma 2.3 implies that $W(\lambda_I) \le CW_I$, where $\le$ means the inequality for quadratic forms, and similarly for W^{-1}, $W^{-1}(\lambda_I) \le C(W^{-1})_I$. By Douglas' Lemma there exist contractions T_1 and T_2 such that

$$W(\lambda_I)^{1/2} = CT_1 {W_I}^{1/2}, \qquad W^{-1}(\lambda_I) = CT_2 {(W^{-1})_I}^{1/2},$$

and therefore

$$\|W(\lambda_I)^{1/2}W^{-1}(\lambda_I)\| \le C^2 \left\| {W_I}^{1/2} {(W^{-1})_I}^{1/2} \right\|$$

$\square$

Lemma 2.5. *If a scalar weight w satisfies the Muckenhoupt (A_2) condition, then*

$$w(\lambda)^{1/2} \le C w^{1/2}(\lambda), \qquad \forall \lambda \in \mathbb{D},$$

where the constant C depends only on Muckenhoupt norm of w.

Proof. Lemma 2.3 implies that the (A_2) condition can be rewritten as

$$\sup_{\lambda\in\mathbb{D}} w(\lambda) w^{-1}(\lambda) < \infty,$$

and the *supremum* depends on Muckenhoupt norm of w. One can write

$$w(\lambda)w^{-1}(\lambda) = \Big[w(\lambda)/\exp\big\{(\log w)(\lambda)\big\}\Big]\Big[w^{-1}(\lambda)/\exp\big\{(\log w^{-1})(\lambda)\big\}\Big],$$

and since expressions in brackets are at least 1 (Jensen inequality, geometric mean less or equal arithmetic) the Muckenhoupt condition is equivalent to the following two:

$$\sup_{\lambda\in\mathbb{D}}\Big[w(\lambda)/\exp\big\{(\log w)(\lambda)\big\}\Big] < \infty, \qquad \sup_{\lambda\in\mathbb{D}}\Big[w^{-1}(\lambda)/\exp\big\{(\log w^{-1})(\lambda)\big\}\Big] < \infty.$$

And by Cauchy–Schwartz and Jensen inequalities

$$\exp\big\{(\log w)(\lambda)\big\} \le \Big(w^{1/2}(\lambda)\Big)^2 \le w(\lambda),$$

that completes the proof. $\square$

Lemma 2.6. *Let W be a $d\times d$ matrix weight satisfying the vector Muckenhoupt condition $(\mathbf{A}_2)$. Then there is a constant C depending on d and the Muckenhoupt norm of W such that*

$$\|W(\lambda)\| \le C\Big(\|W\|^{1/2}(\lambda)\Big)^2, \qquad \forall \lambda \in \mathbb{D},$$

where $\|W\|^{1/2}(\lambda)$ denotes the harmonic extension at the point λ of the defined on the unit circle function $\|W\|^{1/2}$.

Proof. Fix a unit vector $\mathbf{e} \in \mathbb{C}^d$ ($\|\mathbf{e}\| = 1$). As it was shown in the proof of Corollary 2.4 the scalar weight $w = (W\mathbf{e}, \mathbf{e})$ is the Muckenhoupt weight with the same or better Muckenhoupt norm as W. Therefore there exists $C < \infty$ such that

$$(W(\lambda)\mathbf{e}, \mathbf{e}) \le C\left[(W\mathbf{e}, \mathbf{e})^{1/2}(\lambda)\right]^2, \qquad \forall \lambda \in \mathbb{D}$$

If we take a vector $\mathbf{e}$, $\|\mathbf{e}\| = 1$ such that $(W(\lambda)\mathbf{e}, \mathbf{e}) = \|W(\lambda)\|$ we easily get the conclusion of the lemma. □

3. Carleson measure.

For a function f on the circle $\mathbb{T}$ let $f(z)$, $z \in \mathbb{D}$ always denote the harmonic extension of f at z.

Lemma 3.1. *Let W be a harmonic function of n variables with values in the space of strictly positive $d \times d$ matrices ($W(x) = W(x)^* > 0\ \forall x$). Then*

$$\Delta\left(\log(\det W)\right) = -\sum_{j=1}^{n} \operatorname{trace}\left((W^{-1/2}\frac{\partial W}{\partial x_j}W^{-1/2})^2\right)$$

Proof. The proof is a good exercise on multivariate calculus. Using the well known formula for the derivative of a determinant we have

$$\frac{\partial}{\partial x_j}\det W = (\det W)\cdot \operatorname{trace}\left(W^{-1}\frac{\partial W}{\partial x_j}\right),$$

and therefore

$$\frac{\partial}{\partial x_j}(\log\det W) = \operatorname{trace}\left(W^{-1}\frac{\partial W}{\partial x_j}\right).$$

So, using the fact that $(\operatorname{trace} W)' = \operatorname{trace}(W')$ one can get

$$\frac{\partial^2}{\partial {x_j}^2}(\log\det W) = \operatorname{trace}\left(W^{-1}\frac{\partial^2 W}{\partial {x_j}^2}\right) - \operatorname{trace}\left(W^{-1}\frac{\partial W}{\partial x_j}W^{-1}\frac{\partial W}{\partial x_j}\right). \tag{3.1}$$

Here we have also used the fact that

$$\frac{\partial}{\partial x_j}W^{-1} = -W^{-1}\frac{\partial W}{\partial x_j}W^{-1}$$

Since W is a harmonic matrix-valued function,

$$\sum_{j=1}^{n}\operatorname{trace}\left(W^{-1}\frac{\partial^2 W}{\partial {x_j}^2}\right) = 0\,.$$

To complete the proof it is enough to recall that the similarity transformation $A \mapsto TAT^{-1}$ does not change the trace, so

$$\begin{aligned}\Delta\left(\log(\det W)\right) &= -\sum_{j=1}^{n}\operatorname{trace}\left(W^{-1}\frac{\partial W}{\partial x_j}W^{-1}\frac{\partial W}{\partial x_j}\right) = \\ &\quad -\sum_{j=1}^{n}\operatorname{trace}\left(W^{-1/2}\frac{\partial W}{\partial x_j}(W^{-1/2})^2\frac{\partial W}{\partial x_j}W^{-1/2}\right)\end{aligned}$$

□

Theorem 3.2. *If a matrix weight W satisfies the vector Muckenhoupt condition (A_2), then the measures*

$$\left\| W(z)^{-1/2}\left(\frac{\partial}{\partial x}W(z)\right)W(z)^{-1/2}\right\|^2(1-|z|^2)dxdy$$

and

$$\left\| W(z)^{-1/2}\left(\frac{\partial}{\partial y}W(z)\right)W(z)^{-1/2}\right\|^2(1-|z|^2)dxdy$$

are Carleson.

Proof. The vector Muckenhoupt condition (A_2) implies that $W, W^{-1}\in L^1(\mathbb{T})$, therefore $\log(\det W)\in L^1(\mathbb{T})$. Therefore there exists a factorization $W = F^*F$ a.e. on $\mathbb{T}$, where F is an outer function in $H^2(M_{d\times d})$

Since F is an outer function in H^2, $\det F$ is an outer function in $H^{2/d}$. Therefore

$$|\det F(z)| = \exp\{(\log|\det F|)(z)\} = \exp\left\{\frac{1}{2}(\log\det W)(z)\right\} \tag{3.2}$$

It is well known fact that $F^*(z)F(z)\le W(z)$ for any $z\in D$, where $\le$ means the inequality for quadratic forms. There are many proofs of this fact, for example it admits a very simple operator-theoretic interpretation. Explanation that we present here is more function-theoretic: Direct computation shows that

$$\Delta\left(F(z)^*F(z)\right) = 4\left(\bar\partial F(z)^*\right)\left(\partial F(z)\right) = 4\left(\partial F(z)\right)^*\left(\partial F(z)\right)\ge 0\,,$$

so for any $\varepsilon\in\mathbb{C}^d$ the function $\|F(z)e\|^2$ is subharmonic and coincide with $(W(\xi)e,e)$ on $\mathbb{T}$.

We can do the same factorization for W^{-1}. Namely, let G be an outer matrix-valued function in $H^2(M_{d\times d})$ such that $W = G^*G$ on $\mathbb{T}$. We should point to the reader that in general G does not necessarily coincide with F^{-1}. However, applying (3.2) to G one can conclude that

$$|\det G(z)| = \exp\left\{\frac{1}{2}\left(\log\det W^{-1}\right)(z)\right\} = |\det F(z)|^{-1} \tag{3.3}$$

Now we are in position to prove the theorem. As it was shown in Corollary 2.4 the Muckenhoupt condition $(\mathbf{A}_2)$ can be rewritten as

$$\sup_{z\in\mathbb{D}}\left\|W(z)^{1/2}(W^{-1})(z)^{1/2}\right\|<\infty\,. \tag{$\mathbf{A}_2'$}$$

Therefore,

$$\sup_{z\in\mathbb{D}}\left|\det(W(z))\det\left((W^{-1})(z)\right)\right|<\infty\,.$$

Using (3.3) one can rewrite the last inequality as

$$\sup_{z\in\mathbb{D}}\left\{\left[\det W(z)/|\det F(z)|^2\right]\left[\det W^{-1}(z)/|\det G(z)|^2\right]\right\}<\infty$$

Since $F(z)^*F(z)\le W(z)$ and $G(z)^*G(z)\le W^{-1}(z)$, expressions in brackets are at least 1, so, taking into account (3.1)

$$\sup_{z\in\mathbb{D}}\left[\det W(z)/\exp\{(\log\det W)(z)\}\right] = C<\infty\,,$$

or equivalently

$$\log\left\{\det(W(z))\right\} - \left(\log\det W\right)(z) \le C\,, \tag{3.4}$$

where the constant C depends only on the dimension d and the Muckenhoupt constant of W.

By Green's formula and Lemma 3.1

$$\log\left\{\det(W(s))\right\} - \left(\log\det W\right)(s) = -\frac{1}{2\pi}\iint_{\mathbb{D}} \log\left|\frac{1-\overline{s}z}{z-s}\right| \Delta\log\left\{\det(W(z))\right\}\,dxdy =$$

$$= \frac{1}{4\pi}\iint_{\mathbb{D}}\left\{\operatorname{trace}\left(W(z)^{-1/2}\frac{\partial W(z)}{\partial x}W(z)^{-1/2}\right)^2 + \right.$$

$$\left. + \operatorname{trace}\left(W(z)^{-1/2}\frac{\partial W(z)}{\partial y}W(z)^{-1/2}\right)^2\right\}\log\left|\frac{1-\overline{s}z}{z-s}\right|^2\,dxdy$$

Using an elementary inequality $\log(1/a) \ge 1-a$ for $0 < a \le 1$ and the fact that $\|A\| \le \operatorname{trace}A$ for a non-negative matrix A, we can estimate below the last integral by

$$\frac{1}{4\pi}\iint_{\mathbb{D}}\left\|W(z)^{-1/2}\frac{\partial W(z)}{\partial x}W(z)^{-1/2}\right\|^2 \log\left|\frac{1-\overline{s}z}{z-s}\right|^2\,dxdy \ge$$

$$\frac{1}{4\pi}\iint_{\mathbb{D}}\left\|W(z)^{-1/2}\frac{\partial W(z)}{\partial x}W(z)^{-1/2}\right\|^2\left(1-\left|\frac{1-\overline{s}z}{z-s}\right|^2\right)\,dxdy =$$

$$= \iint_{\mathbb{D}}\left\|W(z)^{-1/2}\frac{\partial W(z)}{\partial x}W(z)^{-1/2}\right\|^2 \cdot \frac{(1-|s|^2)(1-|z|^2)}{|1-\overline{s}z|^2}\,dxdy$$

Together with (3.4) this imply

$$\iint_{\mathbb{D}}\frac{(1-|s|^2)}{|1-\overline{s}z|^2}\cdot\left\|W(z)^{-1/2}\frac{\partial W(z)}{\partial x}W(z)^{-1/2}\right\|^2(1-|z|^2)\,dxdy \le C\,, \qquad \forall s\in\mathbb{D},$$

that yields that the measure $\left\|W(z)^{-1/2}\left(\frac{\partial}{\partial x}W(z)\right)W(z)^{-1/2}\right\|^2(1-|z|^2)\,dxdy$ is Carleson.

The measure $\left\|W(z)^{-1/2}\left(\frac{\partial}{\partial y}W(z)\right)W(z)^{-1/2}\right\|^2(1-|z|^2)\,dxdy$ is treated similarly □

4. Equivalence of norms.

Lemma 4.1. *Let a matrix-valued weight W satisfies the vector Muckenhoupt (A_2) condition, and let μ be a Carleson measure. Then for any vector-function $f\in L^2(\mathbb{T},W)$, the following inequality holds,*

$$\iint_{\mathbb{D}}(W(z)f(z),f(z))\,d\mu(z) \le C\int_{\mathbb{T}}(W(\xi)f(\xi),f(\xi))dm(\xi)\,,$$

where the constant C depends the dimension d, the Muckenhoupt constant of W and the Carleson norm of μ.

Proof. Consider an operator $\mathcal{J}: L^2(\mathbb{T})\to L^2(\mu)$ (both L^2 spaces are vector-valued),

$$(\mathcal{J}f)(z) = W(z)^{1/2}\left(W^{-1/2}f\right)(z).$$

To prove the theorem it is enough to show that the operator $\mathcal{J}$ is bounded. We are going to show that its formal adjoint $\mathcal{J}^*$

$$(\mathcal{J}^* f)(\xi) = W^{-1/2}(t)\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{\xi}z|^2} W(z)^{1/2} f(z)\, d\mu(z)$$

is bounded. Direct computations show

$$\begin{aligned}
&\|\mathcal{J}^* f\|^2 = \\
&\int_{\mathbb{D}}\int_{\mathbb{D}}\left[\int_{\mathbb{T}} (W^{-1}(\xi)W(z)^{1/2}f(z), W(s)^{1/2}f(s))\frac{(1-|z|^2)(1-|s|^2)}{|1-\overline{z}\xi|^2|1-\overline{s}\xi|^2}dm(\xi)\right] d\mu(z)d\mu(s) \le \\
&\int_{\mathbb{D}}\int_{\mathbb{D}}\left[\int_{\mathbb{T}} \|W(s)^{1/2}W^{-1}(\xi)W(z)^{1/2}\|\frac{(1-|z|^2)(1-|s|^2)}{|1-\overline{z}\xi|^2|1-\overline{s}\xi|^2}dm(\xi)\right] \|f(z)\|\|f(s)\| d\mu(z)d\mu(s)
\end{aligned}$$

It is easy to estimate inner integral. Namely,

$$|1-\overline{s}z| = |1-\overline{s}\xi + \overline{s}\xi - \overline{s}z| \le |1-\overline{s}\xi| + |s|\cdot|\xi - z| \le |1-\overline{s}\xi| + |1-\overline{z}\xi|\,,$$

so

$$|1-\overline{s}\xi|^{-1}|1-\overline{z}\xi|^{-1} \le |1-\overline{s}z|^{-1}\left(|1-\overline{s}\xi|^{-1} + |1-\overline{z}\xi|^{-1}\right)\,,$$

and finally

$$|1-\overline{s}\xi|^{-2}|1-\overline{z}\xi|^{-2} \le 2\cdot|1-\overline{s}z|^{-2}\left(|1-\overline{s}\xi|^{-2} + |1-\overline{z}\xi|^{-2}\right)\,. \tag{4.1}$$

Assume without loss of generality that $|s| \ge |z|$. Then

$$\begin{aligned}
&\int_{\mathbb{T}} \|W(s)^{1/2}W^{-1}(\xi)W(z)^{1/2}\|\frac{(1-|z|^2)(1-|s|^2)}{|1-\overline{z}\xi|^2|1-\overline{s}\xi|^2}dm(\xi) \le \\
&\frac{2}{|1-\overline{s}z|^2}\int_{\mathbb{T}} \|W(s)^{1/2}W^{-1}(\xi)W(z)^{1/2}\|\left\{\frac{(1-|z|^2)(1-|s|^2)}{|1-\overline{s}\xi|^2} + \right. \\
&\qquad\qquad\left. + \frac{(1-|s|^2)(1-|z|^2)}{|1-\overline{z}\xi|^2}\right\} dm(\xi) \le \\
&\frac{2}{|1-\overline{s}z|^2}\left\{(1-|z|^2)\int_{\mathbb{T}} \|W(s)^{1/2}W^{-1}(\xi)W(s)^{1/2}\|\cdot\|W(s)^{-1/2}W(z)^{1/2}\|\cdot\frac{1-|s|^2}{|1-\overline{s}\xi|^2}dm(\xi) + \right. \\
&\left.(1-|s|^2)\int_{\mathbb{T}} \|W(s)^{1/2}W(z)^{-1/2}\|\cdot\|W(z)^{1/2}W^{-1}(\xi)W(z)^{1/2}\|\cdot\frac{1-|z|^2}{|1-\overline{z}\xi|^2}dm(\xi)\right\}\,.
\end{aligned}$$

The weight W satisfies the Muckenhoupt condition (A_2), hence

$$\int_{\mathbb{T}} \|W(s)^{1/2}W^{-1}(\xi)W(s)^{1/2}\|\cdot\frac{1-|s|^2}{|1-\overline{s}\xi|^2}dm(\xi) \le C < \infty$$

and

$$\int_{\mathbb{T}} \|W(z)^{1/2}W^{-1}(\xi)W(z)^{1/2}\|\cdot\frac{1-|z|^2}{|1-\overline{z}\xi|^2}dm(\xi) \le C < \infty\,,$$

so

$$\int_{\mathbb{T}} \|W(s)^{1/2}W^{-1}(\xi)W(z)^{1/2}\| \frac{(1-|z|^2)(1-|s|^2)}{|1-\overline{z}\xi|^2|1-\overline{s}\xi|^2} dm(\xi) \leq$$
$$C\left\{ \frac{1-|z|^2}{|1-\overline{z}s|^2}\|W(s)^{-1/2}W(z)^{1/2}\| + \frac{1-|s|^2}{|1-\overline{z}s|^2}\|W(s)^{1/2}W(z)^{-1/2}\| \right\}$$

Therefore, to prove the imbedding it is enough to prove that an integral operator with kernel

$$\frac{1-|z|^2}{|1-\overline{z}s|^2}\|W(s)^{-1/2}W(z)^{1/2}\| + \frac{1-|s|^2}{|1-\overline{z}s|^2}\|W(s)^{1/2}W(z)^{-1/2}\|$$

is bounded in $L^2(\mu)$. Since both summands can be obtained from each other by changing s nd z, it is sufficient to estimate only one, i.e. to prove that an integral operator with kernel, say

$$K(z,s) = \frac{1-|s|^2}{|1-\overline{z}s|^2}\|W(s)^{1/2}W(z)^{-1/2}\|$$

is bounded.

Now we are going to apply Senichkin – Vinogradov Test to this integral operator. We need to estimate

$$\int_{\mathbb{D}} K(z,s)K(z,t)d\mu(z) = \int_{\mathbb{D}} \frac{1-|s|^2}{|1-\overline{z}s|^2}\cdot\frac{1-|t|^2}{|1-\overline{z}t|^2}\cdot\|W(s)^{1/2}W(z)^{-1/2}\|\cdot\|W(t)^{1/2}W(z)^{-1/2}\|$$

By (4.1) above

$$|1-\overline{z}s|^{-2}|1-\overline{z}t|^{-2} \leq 2\cdot|1-\overline{s}t|^{-2}\left(|1-\overline{z}s|^{-2}+|1-\overline{z}t|^{-2}\right). \tag{4.2}$$

Product of norms also can be easily estimated:

$$\begin{aligned}
&\|W(s)^{1/2}W(z)^{-1/2}\|\cdot\|W(t)^{1/2}W(z)^{-1/2}\| = \\
&\quad\|W(s)^{1/2}W(z)^{-1/2}\|\cdot\|W(t)^{1/2}W(s)^{-1/2}W(s)^{1/2}W(z)^{-1/2}\| \leq \\
&\quad\|W(s)^{1/2}W(z)^{-1/2}\|\cdot\|W(t)^{1/2}W(s)^{-1/2}\|\cdot\|W(s)^{1/2}W(z)^{-1/2}\| = \\
&\quad= \|W(s)^{1/2}W(z)^{-1}W(s)^{1/2}\|\cdot\|W(t)^{1/2}W(s)^{-1/2}\| \leq \\
&\quad\|W(s)^{1/2}W^{-1}(z)W(s)^{1/2}\|\cdot\|W(t)^{1/2}W(s)^{-1/2}\|,
\end{aligned} \tag{4.3}$$

and similarly

$$\begin{aligned}
&\|W(s)^{1/2}W(z)^{-1/2}\|\cdot\|W(t)^{1/2}W(z)^{-1/2}\| \leq \\
&\quad\|W(t)^{1/2}W^{-1}(z)W(t)^{1/2}\|\cdot\|W(s)^{1/2}W(t)^{-1/2}\|
\end{aligned} \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) together we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} K(z,s)K(z,t)d\mu(z) \leq \\
&2\left\{\frac{1-|t|^2}{|1-\overline{s}t|^2}\|W(t)^{1/2}W(s)^{-1/2}\|\cdot\int_{\mathbb{D}}\frac{1-|s|^2}{|1-\overline{s}z|^2}\|W(s)^{1/2}W^{-1}(z)W(s)^{1/2}\|d\mu(z)+\right.\\
&\left.\frac{1-|s|^2}{|1-\overline{s}t|^2}\|W(s)^{1/2}W(t)^{-1/2}\|\cdot\int_{\mathbb{D}}\frac{1-|t|^2}{|1-\overline{z}t|^2}\|W(t)^{1/2}W^{-1}(z)W(t)^{1/2}\|d\mu(z)\right\}
\end{aligned}$$

If we show that

$$\int_{\mathbb{D}} \frac{1-|s|^2}{|1-\overline{s}z|^2} \|W(s)^{1/2}W^{-1}(z)W(s)^{1/2}\| d\mu(z) \le C < \infty\,, \tag{4.5}$$

where C depends on d, Muckenhoupt constant of W and the Carleson norm of μ, then

$$\int_{\mathbb{D}} K(z,s)K(z,t)d\mu(z) \le C\left(K(s,t)+K(t,s)\right)\,,$$

and we are done.

To prove (4.5) we first recall that by Lemma 2.2 for a fixed $s \in \mathbb{D}$ the weight $V(.) := W(s)^{1/2}W^{-1}(.)W(s)^{1/2}$ is Muckenhoupt and its Muckenhoupt constant is at most the Muckenhoupt constant of W^{-1} (or, equivalently, of W). Let us prove (4.5) for $s=0$: By Lemma 2.6 for $z \in \mathbb{D}$

$$\|W(0)^{1/2}W^{-1}(z)W(0)^{1/2}\| = \|V(z)\| \le C\Big(\|V\|^{1/2}(z)\Big)^2\,;$$

here $\|V\|^{1/2}(z)$ means the Harmonic extension at the point λ of the defined on the circle function $\|V\|^{1/2}$. Since the measure μ is Carleson

$$\int_{\mathbb{D}} \Big(\|V\|^{1/2}(z)\Big)^2 d\mu(z) \le C\int_{\mathbb{T}} \|V(\xi)\| dm(\xi) =$$
$$= C\int_{\mathbb{T}} \Big\|W(0)^{1/2}W^{-1}(\xi)W(0)^{1/2}\Big\|^2 dm(\xi) \le C'\left\|W(0)^{1/2}\Big(\int_{\mathbb{T}} W^{-1}(\xi)dm(\xi)\Big)W(0)^{1/2}\right\| =$$
$$= C'\left\|W(0)^{1/2}W^{-1}(0)W(0)^{1/2}\right\| \le C'' < \infty$$

For $s \in \mathbb{D}$ let φ_s denote the Möbius transformation of the unit disc, $\varphi_s = (s-z)/(1-\overline{s}z)$. Note that

$$\frac{d}{dz}\varphi_s(z) = \frac{1-|s|^2}{(1-\overline{s}z)^2}\,.$$

To prove (4.5) for general s let us fix s and introduce a new variable $\tilde{z} = \varphi_s(z)$ (so $z = \varphi_s(\tilde{z})$, because $\varphi_s \circ \varphi_s$ is an identical mapping) and a new measure $\tilde{\mu}$,

$$\tilde{\mu}(\varphi_s(\Omega)) = \int_{\Omega} |\varphi_s(z)'|\, d\mu(z) = \int_{\Omega} \frac{1-|s|^2}{|1-\overline{s}z|^2} d\mu(z)\,.$$

The measure $\tilde{\mu}$ is Clearly Carleson. Indeed, as it is well known (see [2, 1]) to show that $\tilde{\mu}$ is Carleson it is enough to show that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}\tilde{z}|^2} d\tilde{\mu}(\tilde{z}) < \infty \tag{4.6}$$

By the definition of $\tilde{\mu}$ (recall that $\tilde{z} = \varphi_s(z)$)

$$\int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}\tilde{z}|^2} d\tilde{\mu}(\tilde{z}) = \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}\tilde{z}|^2} |\varphi_s(z)'|\, d\mu(z) = \int_{\mathbb{D}} |\varphi_a(\tilde{z})'| \cdot |\varphi_s(z)'|\, d\mu(z) =$$
$$= \int_{\mathbb{D}} |(\varphi_a(z)\circ\varphi_s(z))'|\, d\mu(z) = \int_{\mathbb{D}} \left|(\varphi_{\varphi_a(s)}(z)`\right| d\mu(z) = \int_{\mathbb{D}} \frac{1-|\varphi_a(s)|^2}{|1-\overline{\varphi_a(s)}z|^2} d\mu(z)\,,$$

and since the measure μ is Carleson (4.6) holds.

Let us also introduce a new weight $\widetilde{W}$, $\widetilde{W} = W \circ \varphi_s$ (equivalently $W = \widetilde{W} \circ \varphi_s$) Clearly for $\tilde{z} = \varphi_s(z)$

$$\widetilde{W}(\tilde{z}) = W(z), \qquad \widetilde{W}^{-1}(\tilde{z}) = W^{-1}(z),$$

so the weight $\widetilde{W}$ satisfies the conformally invariant Muckenhoupt condition $(\mathbf{A}_2')$ (see Corollary 2.4.

Then the left side in (4.5) can be rewritten as

$$\int_{\mathbb{D}} \|\widetilde{W}(0)^{1/2}\widetilde{W}^{-1}(\tilde{z})\widetilde{W}(0)^{1/2}\| d\tilde{\mu}(\tilde{z}) .$$

Since the weight $\widetilde{W}$ is a Muckenhoupt and the measure $\tilde{\mu}$ is Carleson, we reduced the inequality (4.5) to the case $s = 0$ which we have already proved. □

Theorem 4.2. *let W be a matrix Muckenhoupt (A_2) weight, and let f be a vector-function in $L^2(W)$, such that $f(0) = 0$. Then there exists a constant C such that*

$$\frac{1}{C}\int_{\mathbb{T}}(Wf, f)dm \le$$
$$\le \iint_{\mathbb{D}} \left[(W(z)\partial f(z), \partial f(z)) + (W(z)\overline{\partial} f(z), \overline{\partial} f(z))\right] (1 - |z|^2)\, dxdy \le$$
$$\le C \int_{\mathbb{T}}(Wf, f)dm$$

Proof. Applying Green's formula we get

$$\int_{\mathbb{T}}(Wf, f)dm = \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta\left[(W(z)f(z), f(z))\right] \log\frac{1}{|z|}\, dxdy\, .$$

Since for $|z| \le r < 1$ the inequality $|\Delta f(z)| \le C(r)\int_T |f| dm$, the logarithmic singularity $\log\frac{1}{|z|}$ above does not matter and

$$\iint_{\mathbb{D}} \Delta\left[(W(z)f(z), f(z)))\right] \log\frac{1}{|z|}\, dxdy \asymp \iint_{\mathbb{D}} \Delta\left[(W(z)f(z), f(z))\right] (1 - |z|^2)\, dxdy$$

where $\asymp$ means equivalence in a sense of two-sided estimate.

Using the formula $\Delta = 4\partial\overline{\partial} = \overline{\partial}\partial$ we obtain

$$\frac{1}{4}\Delta\left[(Wf, f))\right] = \partial\overline{\partial}\left[(Wf, f))\right] =$$
$$(\partial W\overline{\partial} f, f) + (\partial W f, \partial f) + (\overline{\partial} W\partial f, f) + (\overline{\partial} W f, \overline{\partial} f) + (W\partial f, \partial f) + (W\overline{\partial} f, \overline{\partial} f) =$$
$$2\,\mathrm{Re}(\partial W f, \partial f) + 2\,\mathrm{Re}(\overline{\partial} W f, \overline{\partial} f) + (W\partial f, \partial f) + (W\overline{\partial} f, \overline{\partial} f)$$

Multiplying by $1 - |z|^2$ and integrating over $\mathbb{D}$ we get

$$A^2 := \iint_{\mathbb{D}} \Delta\left[(W(z)f(z), f(z))\right] (1 - |z|^2)\, dxdy =$$
$$8\iint_{\mathbb{D}} \left[\mathrm{Re}(\partial W f, \partial f) + \mathrm{Re}(\overline{\partial} W f, \overline{\partial} f)\right] (1 - |z|^2)\, dxdy +$$
$$4\iint_{\mathbb{D}} \left[(W\partial f, \partial f) + (W\overline{\partial} f, \overline{\partial} f)\right] (1 - |z|^2)\, dxdy \quad (4.7)$$

Denote

$$B^2 := \iint_{\mathbb{D}} \left[(W\partial f, \partial f) + (W\overline{\partial} f, \overline{\partial} f)\right](1-|z|^2)\,dxdy.$$

Note that B^2 is exactly the integral from the statement of the theorem. We already know that $\int_{\mathbb{T}}(Wf,f)dm \asymp A^2$, so to prove the theorem we need to show that $A \asymp B$.

Let us estimate

$$\begin{aligned}
&\left|\iint_{\mathbb{D}} \operatorname{Re}(\partial W f, \partial f)(1-|z|^2)\,dxdy\right| = \\
&\qquad \left|\iint_{\mathbb{D}} \operatorname{Re}(W(z)^{-1/2}\partial W(z) W(z)^{-1/2} W(z)^{1/2} f, W(z)^{1/2}\partial f)(1-|z|^2)\,dxdy\right| \le \\
&\quad \iint_{\mathbb{D}} \left\|W(z)^{-1/2}\partial W(z) W(z)^{-1/2}\right\| \cdot \left\|W(z)^{1/2} f\right\| \cdot \left\|W(z)^{1/2}\partial f\right\|(1-|z|^2)\,dxdy \le \\
&\qquad \left(\iint_{\mathbb{D}} \left\|W(z)^{-1/2}\partial W(z) W(z)^{-1/2}\right\|^2 (W(z)f(z), f(z))(1-|z|^2)\,dxdy\right)^{1/2} \times \\
&\qquad\qquad\qquad \left(\iint_{\mathbb{D}} (W(z)\partial f(z), \partial f(z))(1-|z|^2)\,dxdy\right)^{1/2}
\end{aligned}$$

The second integral in the product is estimated above by B^2.

The first one is also easy to estimate. By Theorem 3.2 the measure

$$\left\|W(z)^{-1/2}\partial W(z) W(z)^{-1/2}\right\|^2 (1-|z|^2)\,dxdy$$

is Carleson, so Lemma 4.1 implies

$$\begin{aligned}
&\iint_{\mathbb{D}} \left\|W(z)^{-1/2}\partial W(z) W(z)^{-1/2}\right\|^2 (W(z)f(z), f(z))(1-|z|^2)\,dxdy \le \\
&\qquad\qquad\qquad \le C\int_{\mathbb{T}}(Wf,f)dm \le C'A^2\,.
\end{aligned}$$

Therefore

$$\left|\iint_{\mathbb{D}} \operatorname{Re}(\partial W f, \partial f)(1-|z|^2)\,dxdy\right| \le C\cdot AB\,,$$

and similarly

$$\left|\iint_{\mathbb{D}} \operatorname{Re}(\overline{\partial} W f, \overline{\partial} f)(1-|z|^2)\,dxdy\right| \le C\cdot AB\,.$$

Summarizing the above one can write

$$A^2 \le C\cdot AB + 4B^2$$

that yields $A \le C'B$. Rewriting (4.7) as

$$4B^2 = A^2 - 8\iint_{\mathbb{D}} \left[\operatorname{Re}(\partial W f, \partial f) + \operatorname{Re}(\overline{\partial} W f, \overline{\partial} f)\right](1-|z|^2)\,dxdy$$

and applying the same trick we obtain $4B^2 \le C\cdot AB + A^2$, so $B \le C''A$. $\square$

5. Boundedness of the Hilbert Transform.

Now it is trivial to prove that if W satisfies the Muckenhoupt (A_2) condition then the Hilbert transform (or, equivalently, the Riesz Projection P_+) bounded operator in $L^2(W)$. We are presenting a formal proof only for the sake of completeness.

Consider a weighted vector-valued L^2-space $\mathcal{H} = L^2(W(z)(1-|z|^2))$ of functions on $\mathbb{D}$ with values in $\mathbb{C}^d$,

$$\|f\|_{\mathcal{H}}^2 = \iint_{\mathbb{D}} (W(z)f(z), f(z))(1-|z|^2)dxdy\,.$$

Let $L_0^2(W) = \{f \in L^2(W) : f(0) = 0\}$. Theorem 4.2 asserts that the mapping

$$f \mapsto \partial f(z) \oplus \overline{\partial} f(z)$$

gives a representation of $L_0^2(W)$ as a subspace of $\mathcal{H} \oplus \mathcal{H}$ (with equivalent norm). Here again $f(z)$, $z \in \mathbb{D}$ denotes the harmonic extension of f.

Consider an operator $\mathbb{P}_+$ on $\mathcal{H}$,

$$\mathbb{P}_+ f \oplus g = f \oplus 0$$

The operator $\mathbb{P}_+$ is obviously bounded, and and its restriction onto $L_0^2(W)$ is nothing but the Riesz Projection P_+. Therefore P_+ is bounded on $L_0^2(W)$. To complete the proof note that for $f \in L^2(W)$

$$P_+ f = P_+(f - f(0)) + f(0)\,.$$

The operator

$$f \mapsto 1 \cdot f(0)$$

is a bounded operator in $L^2(W)$; it was shown above, see the proof of Corollary 2.4, that its norm is exactly $\|W(0)^{1/2} W^{-1}(0)^{-1/2}\|$. Since $f - f(0) \in L_0^2(W)$ the operator P_+ is bounded on $L^2(W)$. □

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