AN OPERATOR CORONA THEOREM.

SERGEI TREIL

Abstract. In this paper some new positive results in the Operator Corona Problem are obtained in rather general situation. The main result is that under some additional assumptions about a bounded analytic operator-valued function F in the unit disc $\mathbb{D}$ the condition

$$F^*(z)F(z) \geq \delta^2 I \qquad \forall z \in \mathbb{D} \qquad (\delta > 0)$$

implies that F has a bounded analytic left inverse. Typical additional assumptions are (any of the following):

(1) *The trace norms of defects $I - F^*(z)F(z)$ are uniformly (in $z \in \mathbb{D}$) bounded. The identity operator I can be replaced by an arbitrary bounded operator here, and F^*F can be changed to FF^*;*
(2) *The function F can be represented as $F = F_0 + F_1$, where F_0 is a bounded analytic operator-valued function with a bounded analytic left inverse, and the Hilbert–Schmidt norms of operators $F_1(z)$ are uniformly (in $z \in \mathbb{D}$) bounded.*

It is now well-known that without any additional assumption, the condition $F^*F \geq \delta^2 I$ is not sufficient for the existence of a bounded analytic left inverse.

Another important result of the paper is the so-called *Tolokonnikov's Lemma* which says that a bounded analytic operator-valued function has an analytic left inverse if and only if it can be represented as a "part" of an invertible bounded analytic function. This result was known for operator-valued function such that the operators $F(z)$ act from a finite-dimensional space, but the general case is new.

Contents

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Notation

$\mathbb{D}$ Open unit disk in the complex plane $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$;

$\mathbb{T}$ Unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$;

$d\mu$ measure on $\mathbb{D}$ defined by $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}dxdy$;

$\partial, \bar\partial$ ∂ and $\bar\partial$-operators, $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\bar\partial = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$;

$\widetilde{\Delta}$ "normalized" Laplacian, $\widetilde{\Delta} = \partial\bar\partial = \frac{1}{4}\Delta = \frac{1}{4}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2})$;

H^2, H^∞ Hardy classes of analytic functions,

$$H^p := \left\{ f \in L^p(\mathbb{T}) : \hat f(k) := \int_{\mathbb{T}} f(z)z^{-k}\frac{|dz|}{2\pi} = 0 \text{ for } k < 0 \right\}.$$

Hardy classes can be identified with spaces of analytic in the unit disk $\mathbb{D}$ functions: in particular, H^∞ is the space of all bounded analytic in $\mathbb{D}$ functions;

$\|\cdot\|$, $|\!|\!|\,.\,|\!|\!|$ norm; since we are dealing with vector- and operator-valued functions, we will use the symbol $\|\,.\,\|$ (usually with a subscript) for the norm in a functions space, while $|\!|\!|\,.\,|\!|\!|$ is used for the norm in the underlying vector (operator) space. Thus for a vector-valued function f the symbol $\|f\|_2$ denotes its L^2-norm, but the symbol $|\!|\!|f|\!|\!|$ stands for the scalar-valued function whose value at a point z is the norm of the vector $f(z)$;

H^2_E vector-valued Hardy class H^2 with values in E;

$L^\infty_{E\to E_*}$ class of bounded functions on the unit circle $\mathbb{T}$ whose values are bounded operators from E to E_*;

$H^\infty_{E\to E_*}$ operator Hardy class of bounded analytic functions whose values are bounded operators from E to E_*;

$$\|F\|_\infty := \sup_{z\in\mathbb{D}} |\!|\!|F(z)|\!|\!| = \operatorname*{esssup}_{\xi\in\mathbb{T}} |\!|\!|F(\xi)|\!|\!|;$$

H_Φ, T_Φ Hankel and Toeplitz operators with symbol Φ.

Throughout the paper all Hilbert spaces are assumed to be separable. We always assume that in any Hilbert space an orthonormal basis is fixed, so any operator $A : E \to E_*$ can be identified with its matrix. Thus besides the usual involution $A \mapsto A^*$ (A^* is the adjoint of A), we have two more: $A \mapsto A^T$ (transpose of the matrix) and $A \mapsto \overline{A}$ (complex conjugation of the matrix), so $A^* = (\overline{A})^T = \overline{A^T}$. Although everything in the paper can be presented in invariant, "coordinate-free" form, use of transposition and complex conjugation makes the notation easier and more transparent.

0. Introduction

The Operator Corona Problem is to find a (preferably local) necessary and sufficient condition for a bounded operator valued function $F \in H^\infty_{E\to E_*}$

to have a left inverse in H^∞, i.e. a function $G \in H^\infty_{E_*\to E}$ such that

$$\text{(B)} \qquad G(z)F(z) \equiv I \qquad \forall z \in \mathbb{D}.$$

Such equations are sometimes called in the literature the Bezout equations, and "B" here is for Bezout. The simplest necessary condition for (B) is

$$\text{(C)} \qquad F^*(z)F(z) \geq \delta^2 I, \qquad \forall z \in \mathbb{D} \qquad (\delta > 0)$$

(the tag "C" is for Carleson).

If the condition (C) implies (B), we say that the Operator Corona Theorem holds.

The Operator Corona Theorem plays an important role in different areas of analysis, in particular in Operator Theory (angles between invariant subspaces, unconditionally convergent spectral decompositions, see [8, 9, 16, 17]), as well as in Control Theory and other applications.

Let us discuss the cases when the Operator Corona Theorem holds.

The first case is $\dim E = 1$, $\dim E_* = n < \infty$. In this case $F = [f_1, f_2, \ldots, f_n]^T$, $G = [g_1, g_2, \ldots, g_n]$ and it is simply the famous Carleson Corona Theorem [4], see also [6, Chapter VIII], [9, Appendix 3].

Later, using the ideas from the T. Wolff's proof of the Carleson Corona Theorem, M. Rosenblum [11], V. Tolokonnikov [14] and Uchiyama [20] independently proved that the Operator Corona Theorem holds if $\dim E = 1$, $\dim E_* = \infty$.

Using simple linear algebra argument, P. Fuhrmann [5] proved that the Operator Corona Theorem holds if $\dim E, \dim E_* < \infty$, and later V. Vasyunin [14][1] extended this result to the case $\dim E^* = \infty$ (but still $\dim E < \infty$).

And finally, a trivial observation: if $F(z)E = E_*$ $\forall z \in \mathbb{D}$, then the left invertibility (C) implies the invertibility of $F(z)$, and so we can simply put $G = F^{-1}$. So in this case the Operator Corona Theorem holds as well.

As for the general Operator Corona Theorem, it was shown earlier by the author ([15], see also [16]) that it fails if $\dim E = +\infty$. Recently it was shown by the author in [18], that the Operator Corona Theorem fails (if $\dim E = \infty$) even if $\operatorname{codim}(F(z)E) = 1$ $\forall z \in \mathbb{D}$, (i.e. if F is very close to the "square" case $F(z)E = E_*$ $\forall z \in \mathbb{D}$, for which the theorem holds)

Note, that for a long time there were no positive results in the infinite-dimensional case ($\dim E = \infty$). Probably the first positive results in this case are the recent results of P. Vitse [21]. She proved that the Operator Corona Theorem holds for operator-valued functions which can be uniformly approximated by *finite* sums $\sum \varphi_k(z)A_k$, where φ_k are *scalar* functions in H^∞ and A_k are *constant* bounded operators. For such functions the image $F(\mathbb{D})$ of the unit disc is a relatively compact (in norm) subset of the space of bounded linear operators, and she also studied different classes of functions for which this condition (relative compactness) alone implies the

[1]It is not a typo, Vasyunin's result was indeed published (with attribution) in the Tolokonnikov's paper.

Operator Corona Theorem.[2] Her technique involved using compactness and the Grothendieck approximation property.

In this paper we prove another group of positive results in infinite-dimensional case. Namely, we show that if an operator-valued function F is a "small" perturbation of a "nice" function F_0 (for example, if F_0 is left invertible in H^∞ and $F - F_0$ belongs to H^∞ with values in the class $\mathfrak{S}_2$ of Hilbert Schmidt operators) then the Operator Corona Theorem holds for such functions

Note, that although this theorem might look like a result obtained from the matrix case using some approximation type reasoning, it is not. While there is some use of approximation in the proof, it is only used as a simple way to justify existence of trace in one of the formulas (see Section 2.1.1 below), and the Matrix Corona Theorem is not used anywhere in the paper.

Moreover, it is probably impossible to get the results using matrix results and approximation! Namely, one can hope to get such result by approximation, if the constants in the matrix Corona Theorem grow not too fast when $\dim E \to \infty$. But as it was recently shown in [18], the norm of the solution grown (at least) exponentially in $\dim E$, so it seems hopeless to use approximation methods.

1. Main results

Let us recall that we say that a function $F \in H^\infty_{E\to E_*}$ is *left invertible in* H^∞ if there exists $G \in H^\infty_{E_*\to E}$ such that $G(z)F(z) \equiv I\ \forall z \in \mathbb{D}$ (and a.e. on $\mathbb{T}$).

Theorem 1.1. *Let $F \in H^\infty_{E\to E_*}$ satisfy **one** of the following conditions:*

(1) *There exist a constant operator A in E such that*
$$\sup_{z\in\mathbb{D}} \| A - F^*(z)F(z) \|_{\mathfrak{S}_1} < \infty;$$

(2) *There exist a constant operator A in E such that*
$$\sup_{z\in\mathbb{D}} \| A - F(z)F^*(z) \|_{\mathfrak{S}_1} < \infty;$$

(3) *There exists a constant operator $B : E \to E_*$ such that*
$$F(z) = B + F_1(z), \qquad \sup_{z\in\mathbb{D}} \| F_1(z) \|_{\mathfrak{S}_2} < \infty.$$

(4) *F can be represented as*
$$F(z) = F_0(z) + F_1(z),$$
where $F_0 \in H^\infty_{E\to E_*}$ is left invertible in H^∞ and $\sup_{z\in\mathbb{D}} \| F_1(z) \|_{\mathfrak{S}_2} < \infty$.

[2]Note, that it is still an open problem whether the relative compactness of $F(\mathbb{D})$ implies that F can be uniformly approximated by functions of form $\sum_{k=1}^N \varphi_k(z)A_k$.

Then for such F the Operator Corona Theorem holds, i.e. the condition

$$F^*(z)F(z) \geq \delta^2 I \qquad \forall z \in \mathbb{D}$$

for some $\delta > 0$ implies that F is left invertible in H^∞.

Theorem 1.2 (Tolokonnikov's Lemma). *A function $F \in H^\infty_{E\to E_*}$ is left invertible in H^∞ if and only if it can be extended to an invertible operator function, i.e. if and only if there exists an auxiliary Hilbert space E_1 and a function $\widetilde{F} \in H^\infty_{E\oplus E_1 \to E_*}$ such that $\widetilde{F}^{-1} \in H^\infty_{E_*\to E\oplus E_1}$ and*

$$\widetilde{F}(z)\big|E = F(z) \qquad \forall z \in \mathbb{D} \quad (\textit{and a.e. on } \mathbb{T})$$

Note, that the existence of $\widetilde{F}$ trivially implies the left invertibility of F.

2. Preliminaries

We will need the following well known theorem.

Recall, that given $\Phi \in L^\infty_{E\to E_*}$, Hankel and Toeplitz operators H_Φ and T_Φ with symbol Φ are defined as

$$\begin{aligned} &H_\Phi : H^2_E \to (H^2_{E_*})^\perp \qquad\qquad && H_\Phi f := P_-(\Phi f)' \\ &T_\Phi : H^2_E \to H^2_{E_*} && T_\Phi f := P_+(\Phi f), \end{aligned}$$

where P_+ and P_- are orthogonal projections onto H^2 and $(H^2)^\perp$ respectively.

Theorem 2.1 (Arveson [2], Sz.-Nagy–Foias [13]). *Let $F \in H^\infty_{E\to E_*}$. The following two statements are equivalent:*

1. *The function F is left invertible in H^∞, i.e. there exists $G \in H^\infty_{E_*\to E}$ such that $GF \equiv I$;*
2. *The Toeplitz operator $T_{\overline{F}}$ is left invertible, that is*

$$\inf_{f\in H^2_E, \|f\|_2=1} \|T_{\overline{F}} f\|_2 =: \delta > 0.$$

Moreover, the best possible norm of a left inverse G is exactly $1/\delta$.

This theorem also can be found in the monograph [8], see Theorem 9.2.1 there.

Note, that this theorem is stated slightly differently in different papers. For example, Theorem 9.2.1 in [8] states that F is *right* invertible in H^∞ if and only if T_{F^*} is left invertible: applying it to F^T we get the statement of Theorem 2.1. Similarly, the theorem in [13] states that F is left invertible in H^∞ if and only if $T_{F^\#}$ is left invertible, where $F^\#(z) := F(\overline{z})$. Again, applying this theorem to $\overline{F(\overline{z})}$ we get Theorem 2.1.

2.1. **Reduction to the H^2 Corona Problem.** According to the above Theorem 2.1, an operator-valued function $F \in H^\infty_{E\to E_*}$ is left invertible in H^∞ if and only if the Toeplitz operator $T_{\overline{F}}$ is left invertible. The latter condition is equivalent to the right invertibility of the adjoint operator $(T_{\overline{F}})^* = T_{\overline{F}^*} = T_{F^T}$.

Since $F^T \in H^\infty_{E_*\to E}$, the Toeplitz operator T_{F^T} is simply the multiplication by F^T. Therefore operator-valued function $F \in H^\infty_{E\to E_*}$ is left invertible in H^∞ if and only if for any $g \in H^2_E$ the equation

$$F^T f = g$$

has a solution $g \in H^2_{E_*}$ satisfying $\|g\|_2 \le C\|f\|_2$ (where the constant C does not depend on g)

The main step in the proof of the main result (Theorem 1.1) is the following theorem giving a sufficient condition for solving the equation $Ff = g$.

Theorem 2.2. *Let $F \in H^\infty_{E_*\to E}$ satisfy*

$$F(z)F^*(z) \ge \delta I \qquad \forall z \in \mathbb{D}$$

for some $\delta > 0$. Let there exist a (real-valued) bounded subharmonic function φ such that its Laplacian $\Delta\varphi$ satisfy

$$\Delta\varphi(z) \ge |\!|\!| F'(z) |\!|\!|^2 \qquad \forall z \in \mathbb{D}.$$

Then for any $g \in H^2_E$ there exists a solution $f \in H^2_{E_*}$ of the equation

$$Ff = g$$

satisfying the estimate $\|f\|_2 \le C\|g\|_2$, where $C = C(\delta, \|\varphi\|_\infty, \|F\|_\infty)$.

2.1.1. *Getting Theorem 1.1 from Theorem 2.2.* As it was discussed above, a function $F \in H^\infty_{E\to E_*}$ is left invertible in H^∞ if and only if for any $g \in H^2_E$ the equation

$$F^T f = g$$

has a solution $f \in H^2_{E_*}$ satisfying $\|f\|_2 \le C\|g\|_2$ (where the constant C does not depend on g).

According to Theorem 2.2 this happens if one can find a subharmonic function φ satisfying

$$\Delta\varphi(z) \ge |\!|\!| (F^T)'(z) |\!|\!|^2 = |\!|\!| F'(z) |\!|\!|^2 \qquad \forall z \in \mathbb{D}.$$

If F satisfies assumption 1 of Theorem 1.1 we put $\varphi(z) = \operatorname{trace}\{A - F^*(z)F(z)\}$ (replacing A by $\operatorname{Re} A = (A + A^*)/2$ if necessary we can assume without loss of generality that $A = A^*$) so

$$\begin{aligned}\Delta\varphi(z) = 4\partial\bar\partial \operatorname{trace}\{A - F^*(z)F(z)\} &= 4 \operatorname{trace}\{(F'(z))^* F'(z)\} \\ &= 4 |\!|\!| F'(z) |\!|\!|^2_{\mathfrak{S}_2} \ge 4 |\!|\!| F'(z) |\!|\!|^2.\end{aligned}$$

Formally, the above reasoning is only a general idea, not a formal proof, because we do not know that the trace appearing during differentiation exists. However it can be easily fixed by using some approximation reasons. For example, the above reasoning works fine if $\dim E < \infty$, so all operators are finite rank ones, so there is no question about trace. So, if P_n is an increasing sequence of orthogonal projections in E, such that $P_n \to I$ strongly as $n \to \infty$, then for $\varphi_n = \text{trace}(P_n(A - F^*F)P_n$

$$\Delta\varphi_n(z) = 4\,\text{trace}\{P_n(F'(z))^*F'(z)P_n\} = 4\,\|F'(z)\|_{\mathfrak{S}_2}^2 \geq 4\,\|F'(z)P_n\|^2,$$

and from here it follows that $\Delta\varphi_n(z)$ is an increasing sequence of subharmonic functions.

Since $\varphi_n \to \varphi$ pointwise as $n \to \infty$, and

$$\|P_nF'(z)\|_{\mathfrak{S}_2}^2 \nearrow \|F'(z)\|_{\mathfrak{S}_2}^2, \qquad \forall z \in \mathbb{D}$$

as $n \to \infty$, one can conclude that indeed

$$\Delta\varphi(z) = 4\,\|F'(z)\|_{\mathfrak{S}_2}^2 \geq 4\,\|F'(z)\|^2.$$

We leave details as an exercise for the reader.

Similarly, for F satisfying assumption 2 of Theorem 1.1 we can define φ as

$$\varphi(z) = tr\{A - F(z)F^*(z)\}.$$

If F satisfies assumption 3 of the theorem, we put

$$\varphi(z) = \text{trace}(F_1^*(z)F_1(z)) = \|F_1(z)\|_{\mathfrak{S}_2}^2.$$

Finally, let us suppose that F satisfies assumption 4 of the theorem. Let us recall that F is represented as $F = F_0 + F_1$, where F_0 is left invertible in H^∞. By Tolokonnikov's Lemma (Theorem 1.2), F_0 can be extended to an invertible (in H^∞) function $\widetilde{F}_0 \in H^\infty_{E\oplus E_1 \to E_*}$. Since $\widetilde{F}_0\Big|E = F$, , we have

$$(\widetilde{F}_0(z))^{-1}F(z) \equiv V,$$

where $V : E \to E \oplus E_1$ is an isometry whose matrix with respect to the decomposition $E \oplus E_1$ is

$$\begin{pmatrix} I \\ \mathbb{O} \end{pmatrix}.$$

Therefore the function $\Phi = \widetilde{F}_0^{-1}F = V + \widetilde{F}_0^{-1}F_1$ satisfies the assumption 3 of the theorem. Clearly,

$$\Phi^*(z)\Phi(z) \geq \delta I, \qquad \forall z \in \mathbb{D} \quad (\delta > 0),$$

so Φ is left invertible in H^∞. Hence, F is also left invertible in H^∞.

3. Proof of Theorem 2.2.

3.1. **Preliminaries.** Our goal is for a given $g \in H^2 := H^2_E$, $\|g\|_2 = 1$ to solve the equation

$$Ff = g, \qquad f \in H^2_{E_*} \tag{3.1}$$

with the estimate $\|f\|_2 \leq C$. By a normal families argument it is enough to suppose that F and g are analytic in a neighborhood of $\mathbb{D}$. Any estimate obtained in this case can be used to find an estimate when F is only analytic on $\mathbb{D}$. Since $F(z)F^*(z) \geq \delta^2 I$, it is easy to find a non-analytic solution f_0 of (3.1),

$$f_0 := \Phi g := F^*(FF^*)^{-1} g.$$

To make f_0 into an analytic solution, we need to find $v \in L^2_{E_*}$ such that $f := f_0 - v \in H^2$ and $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}$. Then

$$Ff = F(f_o - v) = Ff_o - Fv = g,$$

and we are done. The standard way to find such v is to solve a $\overline{\partial}$-equation with the condition $v(z) \in \ker F(z)$ insured by a clever algebraic trick. This trick also admits a "scientific" explanation, for one can get the desired formulas by writing a Koszul complex. What we do in this paper essentially amounts to solving the $\overline{\partial}$-equation $\overline{\partial} v = \overline{\partial} f_0$ on the holomorphic vector bundle $\ker F(z)$. Following the ideas of Matts Andersson [1], which go back to Bo Berndson [3] we use tools from complex differential geometry to solve the corona problem by finding solutions to the $\overline{\partial}$-equation on holomorphic vector bundles.

Since our target audience consists of analysts, all differential geometry will be well hidden. Our main technical tool will be Green's formula

$$\int_{\mathbb{T}} u\, dm - u(0) = \frac{1}{2\pi} \int_{\mathbb{D}} \Delta u \log \frac{1}{|z|} dxdy. \tag{3.2}$$

Instead of the usual Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ it is more convenient for us to use the normalized one $\widetilde{\Delta} := \frac{1}{4}\Delta = \overline{\partial}\partial = \partial\overline{\partial}$. If we denote by μ the measure defined by

$$d\mu = \frac{2}{\pi} \log \frac{1}{|z|} dxdy,$$

then Green's formula can be rewritten as

$$\int_{\mathbb{T}} u\, dm - u(0) = \int_{\mathbb{D}} \widetilde{\Delta} u\, d\mu. \tag{3.3}$$

3.2. **Set-up.** To find the function v we will use duality. We want $f_0 - v \in H^2(E)$, therefore the equality

$$\int_{\mathbb{T}} \langle f_0, h \rangle\, dm = \int_{\mathbb{T}} \langle v, h \rangle\, dm$$

must hold for all $h \in (H^2)^\perp$. Using Green's formula we get

$$\int_{\mathbb{T}} \langle f_0, h \rangle\, dm = \int_{\mathbb{T}} \langle \Phi g, h \rangle\, dm = \int_{\mathbb{D}} \partial\overline{\partial} \left[\langle \Phi g, h \rangle \right]\, d\mu = \int_{\mathbb{D}} \partial \left[\langle \overline{\partial} \Phi g, h \rangle \right]\, d\mu.$$

Here we used the harmonic extension of h, so h is anti-analytic and $h(0) = 0$. The functions $\Phi := F^*(FF^*)^{-1}$ and g are already defined in the unit disc $\mathbb{D}$.

Now the critical moment: let $\Pi(z) := P_{\ker F(z)}$ be the orthogonal projection onto $\ker F(z)$, $\Pi = I - F^* (FF^*)^{-1} F$. Direct computation shows that $\overline{\partial}\Phi = \Pi(\partial\Phi)^*(FF^*)^{-1}$, so $\Pi\overline{\partial}\Phi = \overline{\partial}\Phi$. Therefore, if we define a vector valued function ξ on $\mathbb{D}$ by $\xi(z) := \Pi(z)h(z)$, then

$$\int_{\mathbb{D}} \partial\left[\langle\overline{\partial}\Phi g, h\rangle\right]\,d\mu = \int_{\mathbb{D}} \partial\left[\langle\overline{\partial}\Phi g, \Pi h\rangle\right]\,d\mu = \int_{\mathbb{D}} \partial\left[\langle\overline{\partial}\Phi g, \xi\rangle\right]\,d\mu =: L(\xi).$$

Suppose we are able to prove the estimate

$$|L(\xi)| \le C\|\xi\|_2, \qquad \forall\xi = \Pi h, \quad h \in H^2(E)^\perp. \tag{3.4}$$

Then (by a Hilbert space version of Hahn–Banach Theorem, which is trivial) L can be extended to a bounded linear functional on $L^2(E)$, so there exists a function $v \in L^2(E)$, $\|v\|_2 \le C$, such that

$$L(\xi) = \int_{\mathbb{T}} \langle v, \xi\rangle\,dm, \qquad \forall\xi = \Pi h, \quad h \in H^2(E)^\perp.$$

Replacing v by Πv we can always assume without loss of generality that $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}$. By the construction

$$\int_{\mathbb{T}} \langle v, h\rangle\,dm = \int_{\mathbb{T}} \langle v, \Pi h\rangle\,dm = L(\Pi h) = \int_{\mathbb{T}} \langle \Phi g, h\rangle\,dm \qquad \forall h \in H^2(E)^\perp,$$

so $v - f_0 = v - \Phi g \in H^2(E)$. Therefore, to prove the main theorem we only need to prove the estimate (3.4).

We will need the following lemma, which is proved by direct computations.

Lemma 3.1. *For Π and Φ defined above we have*

$$\begin{aligned}
\partial\Pi &= -F^*\left(FF^*\right)^{-1}F'\Pi \\
\bar{\partial}\Phi &= \Pi\left(F'\right)^*\left(FF^*\right)^{-1} \\
\partial\bar{\partial}\Phi &= \partial\Pi(F')^*(FF^*)^{-1} - (\overline{\partial}\Phi)F'\Phi = \partial\Pi\overline{\partial}\Phi + (\partial\Pi)^*\Phi F'\Phi
\end{aligned}$$

4. Embedding theorems and Carleson measures

As it is well known, Carleson measures play a prominent role in the proof of the Corona theorem, both in Carleson's original proof and in T. Wolff's proof and subsequent modifications. It is also known to the specialists, that essentially all [3] Carleson measures can be obtained from the Laplacian of a bounded subharmonic function. We will need the following well-known theorem, see [9, 8], which was probably first proved by Uchiyama.

Theorem 4.1 (Carleson Embedding Theorem)**.** *Let $\varphi \ge 0$ be a bounded subharmonic function. Then for any $f \in H^2(E)$*

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\cdot \|f(z)\|^2 d\mu(z) \le e\|\varphi\|_\infty \|f\|_2^2.$$

Here $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}dx$, and $\widetilde{\Delta} = \frac{1}{4}\Delta\partial\overline{\partial}$.

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[3] By "essentially all" we mean here that a Carleson measure should first be mollified, to make it smooth, and then it can be obtained from the Laplacian of a subharmonic function.

Proof. Because of homogeneity, we can assume without loss of generality that $\|\varphi\|_\infty = 1$. Direct computation shows that

$$\widetilde{\Delta}\big(e^{\varphi(z)}|f(z)|^2\big) = e^{\varphi}\widetilde{\Delta}\varphi\,|f|^2 + e^{\varphi}\,|\partial\varphi f + \partial f|^2 \geq \widetilde{\Delta}\varphi\cdot|f|^2.$$

Then Green's formula implies

$$\begin{aligned}\int_{\mathbb{D}} \widetilde{\Delta}\varphi\,|f|^2\,d\mu &\leq \int_{\mathbb{D}} \widetilde{\Delta}\big(e^{\varphi}\,|f|^2\big)\,d\mu \\ &= \int_{\mathbb{T}} e^{\varphi}\,|f|^2\,dm - e^{\varphi(0)}\,|f(0)|^2 \leq e\int_{\mathbb{T}} |f|^2\,dm = e\,\|f\|_2^2.\end{aligned}$$

□

Remark 4.2. It is easy to see, that the above Lemma implies the embedding $\int_{\mathbb{D}}|f|^2\,d\mu \leq C\int_{\mathbb{T}}|f|^2\,dm$ (with $C = e$) for all analytic functions f. Using the function $4/(2-\varphi)$ instead of e^{φ} it is possible to get the embedding for harmonic functions with the constant $C = 4$. We suspect the constants e and 4 are the best possible for the analytic and harmonic embedding respectively. We cannot prove that, but it is known that 4 is the best constant in the dyadic (martingale) Carleson Embedding Theorem.

We will need a similar embedding theorem for functions of form $\xi = \Pi h$, $h \in H^2(E)^\perp$. Such functions are not harmonic, so the Carleson Embedding Theorem does not apply. As a result, the proof is more complicated, and the constant is significantly worse. We will need several formulas. Recall that $\Pi(z) = P_{\ker F(z)}$ is the orthogonal projection onto $\ker F(z)$, $\Pi = I - F^*(FF^*)^{-1}F$, and that $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}dxdx$.

Lemma 4.3. *Let $\varphi \geq 0$ be a bounded subharmonic function in $\mathbb{D}$ satisfying*

$$\widetilde{\Delta}\varphi(z) \geq |\partial\Pi(z)|^2, \qquad \forall z \in \mathbb{D},$$

and let $K = \|\varphi\|_\infty$. Then for all ξ of form $\xi = \Pi h$, $h \in H^2(E)^\perp$

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\,|\xi(z)|^2\,d\mu(z) \leq eKe^K\|\xi\|_2^2$$

and

$$\int_{\mathbb{D}} |\overline{\partial}\xi|^2\,d\mu \leq (1 + eKe^K)\|\xi\|_2^2.$$

Proof. Let us take arbitrary subharmonic $\varphi \geq 0$ and compute $\widetilde{\Delta}\left(e^{\varphi}\,|\xi|^2\right)$. Lemma 3.1 implies that $\Pi\partial\Pi = 0$ and $\partial\Pi\Pi = \partial\Pi$. Therefore, using $\partial h = 0$ we get $\partial\xi = \partial\left(\Pi h\right) = \partial\Pi h + \Pi\partial h = \partial\Pi h = \partial\Pi\xi$, and so

$$\langle\partial\xi, \xi\rangle = \langle\partial\xi, \Pi\xi\rangle = \langle\partial\Pi\xi, \Pi\xi\rangle = 0.$$

Therefore

$$\partial\left(e^{\varphi}|\xi|^2\right) = e^{\varphi}\partial\varphi\,|\xi|^2 + e^{\varphi}\langle\partial\xi,\xi\rangle + e^{\varphi}\langle\xi,\overline{\partial}\xi\rangle = e^{\varphi}\partial\varphi\,|\xi|^2 + e^{\varphi}\langle\xi,\overline{\partial}\xi\rangle.$$

Taking $\overline{\partial}$ of this equality (and using again $\langle \xi, \partial\xi\rangle = 0$) we get

$$\widetilde{\Delta}\left(e^{\varphi}\,|\xi|^2\right) = e^{\varphi}\left(\widetilde{\Delta}\varphi\,|\xi|^2 + |\overline{\partial}\varphi\xi + \overline{\partial}\xi|^2 + \langle \xi, \widetilde{\Delta}\xi\rangle\right).$$

To handle $\langle \xi, \widetilde{\Delta}\xi\rangle$ we take the ∂ derivative of the equation $\langle \xi, \partial\xi\rangle = 0$ to get

$$\langle \partial\xi, \partial\xi\rangle - \langle \xi, \overline{\partial}\partial\xi\rangle = 0,$$

and therefore $\langle \xi, \widetilde{\Delta}\xi\rangle = -|\partial\xi|^2 = -|(\partial\Pi)\xi|^2$. Since $\varphi \ge 0$

$$\int_{\mathbb{D}}\left(\widetilde{\Delta}\varphi\,|\xi|^2 - |(\partial\Pi)\xi|^2\right)d\mu \le$$

$$\int_{\mathbb{D}}\left(\widetilde{\Delta}\varphi\,|\xi|^2 - |(\partial\Pi)\xi|^2 + |\bar{\partial}\varphi\xi + \bar{\partial}\xi|^2\right)e^{\varphi}d\mu = \int_{\mathbb{T}} e^{\varphi}|\xi|^2 dm; \tag{4.1}$$

the equality is just Green's formula (recall that $\xi(0) = 0$). In the last inequality replacing φ by $t\varphi$, $t > 1$ we get

$$\int_{\mathbb{D}}\left(t\widetilde{\Delta}\varphi\,|\xi|^2 - |(\partial\Pi)\xi|^2\right)\,d\mu \le \int_{\mathbb{T}} e^{t\varphi}\,|\xi|^2 dm \le e^{tK}\|\xi\|_2^2$$

Now we use the inequality $\widetilde{\Delta}\varphi \ge |\partial\Pi|^2$. It implies $\widetilde{\Delta}\varphi\,|\xi|^2 - |\partial\Pi\xi|^2 \ge 0$, and therefore

$$(t-1)\int_{\mathbb{D}}\widetilde{\Delta}\varphi\,|\xi|^2 d\mu \le e^{tK}\|\xi\|_2^2.$$

Hence

$$\int_{\mathbb{D}}\widetilde{\Delta}\varphi\,|\xi|^2 d\mu \le \min_{t>1}\frac{e^{tK}}{t-1}\,\|\xi\|_2^2 = eKe^{K}\|\xi\|_2^2$$

(minimum is attained at $t = 1 + 1/K$), and thus the first statement of the lemma is proved.

To prove the second statement, put $\varphi \equiv 0$ in (4.1) (we do not use any properties of φ except that $\varphi \ge 0$ in (4.1)) to get

$$\int_{\mathbb{D}}\left(|\overline{\partial}\xi|^2 - |(\partial\Pi)\xi|^2\right)\,d\mu = \int_{\mathbb{T}}|\xi|^2\,dm = \|\xi\|_2^2.$$

But the second term can be estimated

$$\int_{\mathbb{D}}|(\partial\Pi)\xi|^2\,d\mu \le \int_{\mathbb{D}}\widetilde{\Delta}\varphi\,|\xi|^2\,d\mu \le eKe^{K}$$

and therefore $\int_{\mathbb{D}}|\overline{\partial}\xi|^2\,d\mu \le (1 + eKe^{K})\cdot\|\xi\|_2^2$. □

5. Main estimates

As it was already discussed before, to prove the main result (Theorem 1.1) we need to prove Theorem 2.2, the proof of which is (see Section 3) is reduced to the estimate (3.4), i.e. the estimate

$$|L(\xi)| = \left|\int_{\mathbb{D}}\partial\left[\langle\overline{\partial}\Phi g, \xi\rangle\right]\,d\mu\right| \le C\|\xi\|_2, \qquad \forall \xi = \Pi h, \quad h \in H^2(E)^{\perp};$$

here, recall $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}dxdy$. Computing ∂ of the inner product we get

$$\begin{aligned} L(\xi) &= \int_{\mathbb{D}} \partial\left[\langle\bar\partial\Phi g, \xi\rangle\right] d\mu \\ &= \int_{\mathbb{D}} \langle\partial\bar\partial\Phi g, \xi\rangle d\mu + \int_{\mathbb{D}} \langle\bar\partial\Phi g', \xi\rangle d\mu + \int_{\mathbb{D}} \langle\bar\partial\Phi g, \bar\partial\xi\rangle d\mu \\ &= I + II + III. \end{aligned}$$

The assumption of Theorem 2.2 is that there exists a real-valued function $\varphi \in L^\infty(\mathbb{D})$, satisfying $\widetilde\Delta\varphi(z)\| \ge \|F'(z)\|^2$, $\forall z \in \mathbb{D}$. Note, that without loss of generality we can assume that $\varphi \ge 0$. Since (see Lemma 3.1) $\partial\Pi = -F^*\left(FF^*\right)^{-1}F'\Pi$ we can conclude that $\|\partial\Pi\|^2 \le A|\widetilde\Delta\varphi|$,

To estimate the first integral recall that $\partial\bar\partial\Phi = \partial\Pi\bar\partial\Phi + (\partial\Pi)^*\Phi F'\Phi$. We get

$$I = \int_{\mathbb{D}} \langle\partial\bar\partial\Phi g, \xi\rangle d\mu = \int_{\mathbb{D}} \left\{\langle\partial\Pi\bar\partial\Phi g, \xi\rangle + \langle(\partial\Pi)^*\Phi F'\Phi g, \xi\rangle\right\} d\mu.$$

Since $(\partial\Pi)^*\Pi = 0$ we have $(\partial\Pi)^*\xi = 0$, and so $\langle\partial\Pi\bar\partial\Phi g, \xi\rangle = 0$. Therefore

$$I = \int_{\mathbb{D}} \langle(\partial\Pi)^*\Phi F'\Phi g, \xi\rangle d\mu = \int_{\mathbb{D}} \langle\Phi F'\Phi g, (\partial\Pi)\xi\rangle d\mu,$$

and the Cauchy–Schwarz inequality implies

$$\begin{aligned} |I| &\le \left(\int_{\mathbb{D}} \|\Phi F'\Phi g\|^2 d\mu\right)^{1/2} \left(\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2 d\mu\right)^{1/2} \\ &\le \left(\int_{\mathbb{D}} \|\Phi\|^4 \|F'\|^2 \|g\|^2 d\mu\right)^{1/2} \left(\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2 d\mu\right)^{1/2} \end{aligned}$$

By the Carleson embedding theorem (Theorem 4.1)

$$\int_{\mathbb{D}} \|\Phi\|^4 \, \|F'\|^2 \, \|g\|^2 d\mu \le \|\Phi\|_\infty^4 e\|\varphi\|_\infty \|g\|_2^2,$$

and by Lemma 4.3

$$\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2 d\mu \le C\|\xi\|_2^2,$$

so $|I| \le C\|g\|_2\|\xi\|_2$.

Let us estimate II:

$$|II| \le \int_{\mathbb{D}} \|g'\| \cdot \|\bar\partial\Phi\| \cdot \|\xi\| \, d\mu \le \left(\int_{\mathbb{D}} \|g'\|^2 d\mu\right)^{1/2} \left(\int_{\mathbb{D}} \|\bar\partial\Phi\|^2 \|\xi\|^2 \, d\mu\right)^{1/2}.$$

We know that $\int_{\mathbb{D}} \|g'\|^2 d\mu = \|g\|_2^2 - |g(0)|^2 \le \|g\|_2^2$. Using the fact that $\|\bar\partial\Phi\|^2 \le C\|F'\|^2 \le C\Delta\varphi$, we get using Lemma 4.3

$$\int_{\mathbb{D}} \|\bar\partial\Phi\|^2 \|\xi\|^2 \, d\mu \le C\int_{\mathbb{D}} \Delta\varphi \|\xi\|^2 \, d\mu \le C'\|\xi\|_2^2,$$

so again we have the desired estimate for II, $|II| \le C\|g\|_2\|\xi\|_2$.

The last term is estimated similarly to the first one, only simpler:

$$|III| \le \int_{\mathbb{D}} \|\bar\partial\Phi\| \cdot \|g\| \cdot \|\bar\partial\xi\| \, d\mu \le \left(\int_{\mathbb{D}} \|\bar\partial\Phi\|^2 \|g\|^2 d\mu\right)^{1/2} \left(\int_{\mathbb{D}} \|\bar\partial\xi\|^2 \, d\mu\right)^{1/2}.$$

Again, $\|\bar\partial\Phi\|^2 \le C\widetilde\Delta\varphi$, and so by Theorem 4.1

$$\int_{\mathbb{D}} \|\bar\partial\Phi\|^2 \|g\|^2 d\mu \le C \int_{\mathbb{D}} \widetilde\Delta\varphi \; \|g\|^2 \, d\mu \le C' \|g\|_2^2.$$

And Lemma 4.3 implies that

$$\int_{\mathbb{D}} \|\bar\partial\xi\|^2 \, d\mu \le C \|\xi\|_2^2$$

So, Theorem 2.2 is proved.

6. Proof of Tolokonnikov's Lemma (Theorem 1.2)

To prove the Tolokonnikov's Lemma, we need the following surprising but simple result due to N. Nikolski (personal communication).

Lemma 6.1. *Let $F \in H^\infty_{E\to E_*}$ satisfies*

$$F^*(z)F(z) \ge \delta^2 I, \qquad \forall z \in \mathbb{D}.$$

Then F is left invertible in $H^\infty_{E\to E_*}$ (i.e. there exists $G \in H^\infty_{E_*\to E}$ such that $GF \equiv I$) if and only if there exists a function $\mathcal{P} \in H^\infty_{E_*\to E_*}$ whose values are projections (not necessarily orthogonal) onto $F(z)E$ for all $z \in \mathbb{D}$ (and a.e. on $\mathbb{T}$).

Remark 6.2. Note that the condition

$$\mathcal{P}(\xi)E_* = F(z)E \qquad \text{a.e. on } \mathbb{T} \tag{6.1}$$

(together with all other assumptions of the lemma except the assumption that (6.1) holds for all $z \in \mathbb{D}$) is not sufficient for the left invertibility of F in H^∞.

Indeed, in [18] a function $F \in H^\infty_{E\to E}$ was constructed, which satisfies $F^*F \ge \delta^2 I$ and $F(\xi)E = E$ a.e. on $\mathbb{T}$, but is not left invertible in H^∞. Treating this function as a function in $H^\infty_{E\to E\oplus E_1}$, where E_1 is an auxiliary Hilbert space, one can see that the function $\mathcal{P} \in H^\infty_{E\to E\oplus E_1}$, $\mathcal{P}(z) \equiv P_E$ satisfies (6.1), but F is still not left invertible in H^∞

Proof of Lemma 6.1. Let F be left invertible in H^∞, and let G be one of its left inverses. Define $\mathcal{P} \in H^\infty_{E_*\to E_*}$ by

$$\mathcal{P}(z) = F(z)G(z).$$

Direct computation shows that $\mathcal{P}^2 = \mathcal{P}$, so the values of $\mathcal{P}$ are projections.

Since $GF \equiv I$,

$$G(z)E_* = E \qquad \forall z \in \mathbb{D} \text{ and a.e. on } \mathbb{T}.$$

Therefore

$$\mathcal{P}(z)E_* = F(z)G(z)E_* = F(z)E \qquad \text{a.e. on } \mathbb{T} \text{ and } \forall z \in \mathbb{D},$$

i.e. $\mathcal{P}(z)$ is indeed a projection onto $F(z)E$.

Suppose now that there exists a projection-valued function $\mathcal{P} \in H^\infty_{E_*\to E_*}$, whose values are projections onto $F(z)E$ for all $z \in \mathbb{D}$. We want to show that F is left invertible in H^∞.

First of all let us notice that locally, in a neighborhood of each point $z_0 \in \mathbb{D}$ the function $F(z)$ has analytic left inverse. Indeed, if an operator $G_0 : E_* \to E$ is a constant left inverse to the operator $F(z_0)$, i.e. if $G_0 F(z_0) = I$, then

$$G_0 F(z) = I - G_0 \cdot (F(z_0) - F(z)),$$

so the inverse of $G_0 F(z)$ is given by the analytic function

$$A(z) := \sum_{k=0}^{\infty} [G_0 \cdot (F(z_0) - F(z))]^k$$

defined in a neighborhood of z_0. So, $A(z)G_0$ is a local analytic inverse of $F(z)$.

Since (for a fixed $z \in D$) the operator $F(z)$ is left invertible, it is invertible if we treat it as an operator from E to $F(z)E$. Let $F^\dagger(z) : F(z)E \to E$ be the inverse of such "restricted" $F(z)$.

Note, that for any (not necessarily analytic) left inverse $\widetilde{G}(z)$ of $F(z)$

$$\widetilde{G}(z) \bigm| F(z)E = F^\dagger(z) \bigm| F(z)E. \tag{6.2}$$

Since $\mathcal{P}(z)$ is a projection onto $F(z)E$, the function G,

$$G(z) := F^\dagger(z)\mathcal{P}(z)$$

is well defined and bounded (since both $F^\dagger$ and $\mathcal{P}$ are bounded). It is easy to see that $G(z)F(z) \equiv I$, so to complete the proof one needs only to show that G is analytic.

Fix a point $z_0 \in \mathbb{D}$ and let $G_{z_0}(z)$ be a *local* analytic left inverse of $F(z)$ defined in a neighborhood of z_0. It follows from (6.2) that

$$G(z) = F^\dagger(z)\mathcal{P}(z) = G_{z_0}(z)$$

in a neighborhood of z_0, so $G(z)$ is analytic there. Since z_0 is arbitrary, G is analytic in $\mathbb{D}$. □

Proof of Tolokonnikov's Lemma. Let $F \in H^\infty_{E\to E_*}$ be left invertible in H^∞, and let $\mathcal{P} \in H^\infty_{E_*\to E_*}$ be a projection-valued function from Lemma 6.1 satisfying

$$\mathcal{P}(z)E_* = F(z)E \qquad \forall z \in \mathbb{D}.$$

Define a projection-valued function $\mathcal{Q} \in H^\infty_{E_*\to E_*}$ by $\mathcal{Q}(z) := I - \mathcal{P}(z)$, $z \in \mathbb{D}$, and let

$$\mathcal{Q} = \Theta R, \qquad \Theta \in H^\infty_{E_1\to E_*} \text{ is inner, } R \in H^\infty_{E_*\to E_1} \text{ is outer,}$$

be its inner-outer factorization; here E_1 is an auxiliary Hilbert space.

Since the multiplication by $\mathcal{Q}$ is a bounded projection in $H^2_{E_*}$, the set

$$\mathcal{E} := \mathcal{Q}H^2_{E_*} = \{f \in H^2_{E_*} : f = \mathcal{Q}g, g \in H^2_{E_*}\}$$

is a closed subspace of $H^2_{E_*}$. Therefore, by properties of inner-outer factorization

$$\Theta H^2_{E_1} = \operatorname{clos} \mathcal{Q} H^2_{E_*} = \mathcal{Q} H^2_{E_*} = \mathcal{E},$$

so $\mathcal{E} = \Theta H^2_{E_1}$. In particular, this implies that $\Theta(z)E_1 = \mathcal{Q}(z)E_*$ for all $z \in \mathbb{D}$.

Let us show that $\ker \Theta(z) = \{0\}$ for all $z \in \mathbb{D}$. Indeed, suppose for some $z_0 \in \mathbb{D}$ and $e \in E_1$ we have $\Theta(z_0)e = 0$. Then for $f \in \mathcal{E}$ defined by $f(z) = \Theta(z)e$ we have $f(z_0) = 0$. Then $f_1 = f/(z - z_0)$ is in $H^2_{E_*}$ and therefore in $\mathcal{E}$. On the other hand $g = (z - z_0)^{-1}e$ is the only $L^2_{E_1}$ solution of

$$\Theta(\xi)g(\xi) = f_1(\xi) = f(\xi)/(\xi - z_0) \qquad \text{a.e. on } \mathbb{T},$$

and $g \notin H^2_{E_1}$. Therefore $\mathcal{E} \neq \Theta H^2_{E_1}$ and we got a contradiction.

Define $\widetilde{F} \in H^\infty_{E \oplus E_1 \to E_*}$ by $\widetilde{F} := F \oplus \Theta$, meaning that

$$\widetilde{F}(z)e \oplus e_1 = F(z)e + \Theta(z)e_1, \qquad e \in E, e_1 \in E_1.$$

Since for any fixed $z \in \mathbb{D}$ the subspaces $F(z)E = \mathcal{P}(z)E_*$ and $\Theta(z)E_1 = \mathcal{Q}(z)E_*$ are complimentary subspaces (in particular, they have trivial intersection), and operators $F(z)$ and $\Theta(z)$ have trivial kernels, we have

$$\ker \widetilde{F}(z) = \{0\} \qquad \forall z \in D.$$

Direct computation shows that $\widetilde{G} \in H^\infty_{E_* \to E \oplus E_1}$ defined by

$$\widetilde{G}(z)e = F^{-1}(z)\mathcal{P}(z)e \oplus R(z)e, \qquad z \in D, \quad e \in E_*$$

is a right inverse for $\widetilde{F}$. Since $\ker \widetilde{F}(z) = \{0\}$ for all $z \in D$, we can conclude that $\widetilde{F}$ is invertible in H^∞. □

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Department of Mathematics, Brown University, 151 Thayer Str./Box 1917, Providence, RI 02912, USA

E-mail address: `treil@math.brown.edu`

URL: `http://www.math.brown.edu/~treil`