ESTIMATES IN THE CORONA THEOREM AND IDEALS OF H^∞: A PROBLEM OF T. WOLFF

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Abstract. The main result of the paper is that there exist functions $f_1, f_2, f \in H^\infty$ satisfying the "Corona condition"

$$|f_1(z)| + |f_2(z)| \geqslant |f(z)| \qquad \forall z \in \mathbb{D},$$

and such, that f^2 does not belong to the ideal $\mathcal{I}$ generated by f_1, f_2, i. e. f^2 cannot be represented as $f^2 \equiv f_1 g_1 + f_2 g_2$, $g_1, g_2 \in H^\infty$. This gives a negative answer to an old question by T. Wolff [10].

Note, that it was well known before that under the same assumptions f^p belongs to the ideal if $p > 2$, but a counterexample can be constructed for $p < 2$, so our case $p = 2$ is a critical one.

To get the main result we improved lower estimates for the solution of the Corona problem. Namely, we proved that given $\delta > 0$ there exist finite Blaschke products f_1, f_2 satisfying the Corona condition

$$|f_1(z)| + |f_2(z)| \geqslant \delta \qquad \forall z \in \mathbb{D},$$

and such, that for any $g_1, g_2 \in H^\infty$ satisfying $f_1 g_1 + f_2 g_2 \equiv 1$ (solution of the Corona problem), the estimate $\|g_1\|_\infty \geqslant C\delta^{-2}\log(-\log\delta)$ holds. The estimate $\|g_1\|_\infty \geqslant C\delta^{-2}$ was obtained earlier by V. Tolokonnikov.

Contents

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Notation

$:=$	equal by definition;
$\mathbb{C}$	the complex plane;
$\mathbb{D}$	the unit disc, $\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert < 1\}$;
$\mathbb{T}$	unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : \lvert z\rvert = 1\}$;
H^1, H^∞	Hardy spaces: $H^p := \{f \in L^p(T) : f(z) = \sum_0^\infty a_k z^k\}$; spaces H^p can be naturally identified with the spaces of analytic functions on the disc $\mathbb{D}$. In particular, H^∞ consists of all bounded analytic function on the unit disc $\mathbb{D}$ (with the *supremum* norm);

1. Introduction and main results

1.1. **Ideals of H^∞.** The famous Carleson Corona Theorem states that if functions $f_1, f_2, \dots, f_n \in H^\infty$ satisfy

$$|f_1(z)| + |f_2(z)| + \ldots + |f_n(z)| \geqslant 1 \qquad \forall z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

then 1 belongs to the ideal generated by functions $f_1, f_2, \dots, f_n$, i. e. there exist functions $g_1, g_2, \dots, g_n \in H^\infty$ such that

$$f_1 g_1 + f_2 g_2 + \ldots + f_n g_n \equiv 1.$$

One can try to generalize this result by replacing 1 by an arbitrary function $f \in H^\infty$. Namely, one can ask, does the condition

$$|f_1(z)| + |f_2(z)| + \ldots + |f_n(z)| \geqslant |f(z)| \qquad \forall z \in \mathbb{D} \tag{1.1}$$

implies that f belongs to the ideal $\mathcal{I}$ generated by $f_1, f_2, \dots, f_n$, i. e. that there exist functions $g_1, g_2, \dots, g_n \in H^\infty$ such that

$$f_1 g_1 + f_2 g_2 + \ldots + f_n g_n \equiv f.$$

Note, that the above condition (1.1) is clearly necessary.

An example by Rao [7], (see also [4], Chapter VIII) shows that the answer is negative (if one excludes the trivial case $n = 1$).

A natural question appears, if some power of f belongs to the ideal $\mathcal{I}$.

Problem A. *Let $n \geqslant 2$ and $p \geqslant 1$ be fixed, and let $f_1, f_2, \dots, f_n, f \in H^\infty$. Does the condition* (1.1) *imply that f^p belongs to the ideal $\mathcal{I}$ generated by $f_1, f_2, \dots, f_n$, i. e. that there exist functions $g_1, g_2, \dots, g_n \in H^\infty$ such that*

$$f_1(z)g_1(z) + f_2(z)g_2(z) + \ldots + f_n(z)g_n(z) \equiv f(z)^p.$$

Note, that if f does not have zeroes in the unit disc $\mathbb{D}$, we don't have to assume that the exponent p is integer.

There is another natural question to ask:

Problem B. *Let $n \geqslant 2$ and $p \geqslant 1$ be fixed, and let $f_1, f_2, \dots, f_n, f \in H^\infty$. Does the condition*

$$\big(|f_1(z)| + |f_2(z)| + \ldots + |f_n(z)|\big)^p \geqslant |f(z)| \qquad \forall z \in \mathbb{D}$$

imply that f belongs to the ideal $\mathcal{I}$ generated by $f_1, f_2, \dots, f_n$?

Clearly, an affirmative answer to Problem B for some n and p implies the affirmative answer to the Problem A (for the same n and p).

Problem B is a particular case of the following more general problem.

Let h be a continuous increasing (in a neighborhood of 0) function on $[0, \infty)$.

Problem C. *Let $n \geqslant 2$. Suppose functions $f_1, f_2, \dots, f_n, f \in H^\infty$ satisfy*

$$h\big(|f_1(z)| + |f_2(z)| + \ldots + |f_n(z)|\big) \geqslant |f(z)| \qquad \forall z \in \mathbb{D}$$

For which functions h, this condition implies that f belongs to the ideal $\mathcal{I}$ generated by $f_1, f_2, \dots, f_n$?

It is known that the answers to both Problem A and Problem B are affirmative for $p > 2$ and are both negative for $p < 2$.

Let us also mention a result by U. Cegrell, that Problem C has an affirmative answer for $h(s) = s^2 / \big((-\log s)^{3/2}(\log(-\log s))^{3/2}\log(\log(-\log s))\big)$.

Let us also mention a result by J. Bourgain [1], that if $\lim_{t\to 0} h(t)/t = 0$, then condition (1.1) implies that f belongs to the norm closure of the ideal $\mathcal{I}$. It was also shown in [1] that this statement does not hold for $h(t) \equiv t$.

Problems A and B for $p = 2$ had remained open. In the famous problem book [5] T. Wolff [10] posed a question about Problem A for $p = 2$.

Main result of the present paper is that the answer is negative. Namely, the following theorem holds.

Theorem 1.1. *There exist functions $f_1, f_2, f \in H^\infty$, such that*

$$|f_1(z)| + |f_2(z)| \geqslant |f(z)| \qquad \forall z \in \mathbb{D},$$

but f^2 does not belong to the ideal $\mathcal{I}$ generated by f_1, f_2, i. e. there are no functions $g_1, g_2 \in H^\infty$ such that $f^2 \equiv f_1 g_1 + f_2 g_2$.

Note, that the above result implies that Problem B for $p = 2$ also has negative answer.

1.2. **Estimates in the Corona Theorem.** Suppose functions $f_1, f_2, \dots, f_n \in H^\infty$ satisfy the estimates

$$1 \geqslant \big(|f_1(z)|^2 + |f_2(z)|^2 + \ldots + |f_n(z)|^2\big)^{1/2} \geqslant \delta > 0 \qquad \forall z \in \mathbb{D}. \tag{1.2}$$

According to the Carleson Corona Theorem, there exist functions $g_1, g_2, \dots, g_n \in H^\infty$ such that $f_1g_1 + f_2g_2 + \ldots + f_ng_n \equiv 1$. We are interested in estimates of the norms $\|g_k\|_\infty$ of the solution of the Corona Problem.

The best upper estimate known is,

$$\Big(\sum_k |g_k(z)|^2\Big)^{1/2} \leqslant \frac{C}{\delta^2}(-\log\delta) \qquad \forall z \in \mathbb{D}.$$

where C is an absolute constant (not depending on n). It means that for any corona data satisfying (1.2) one can find a solution satisfying the above estimates (solution of the Corona Problem is clearly not unique). This estimate was first proved by Uchiyama [9], see also [6] for another proof.

A slightly worse estimate $\left(\sum_k |g_k(z)|^2\right)^{1/2} \leqslant \frac{C}{\delta^2}(-\log\delta)^{3/2}$ was abtained inependently by V. Tolokonnikov [8].

Note, that if we do not care about the constant C, and don't care if it depends on n, we can replace condition (1.2) by the following one

$$1 \geqslant |f_1(z)| + |f_2(z)| + \ldots + |f_n(z)| \geqslant \delta > 0 \qquad \forall z \in \mathbb{D}. \tag{1.3}$$

It was also shown by Tolokonnikov in [8], that it is impossible to get a better estimate than C/δ^2. Namely, he constructed functions $f_1, f_2 \in H^\infty$ satisfying (1.3) such that for any solution $g_1, g_2 \in H^\infty$ of the Corona Problem ($f_1 g_1 + f_2 g_2 \equiv 1$) the inequality $\|g_1\|_\infty \geqslant c/\delta^2$ holds.

To prove the main result of the paper we will need a better lower bound.

Before stating the theorem, let us remind the reader that a *finite Blaschke product* is a function B that can be represented as $B(z) = c \cdot \prod_1^n b_{\lambda_k}(z)$, where $c \in \mathbb{T}$, $b_\lambda := (z-\lambda)/(1-\overline{\lambda} z)$, $|\lambda| < 1$ ($b_0 = z$), and $\lambda_k \in \mathbb{D}$. The function b_λ is called *Blaschke factor*, or Möbius transform.

An equivalent description is that B is a rational function with poles outside of the closed unit disk and unimodular on the unit circle $\mathbb{T}$ (i. e. satisfying $|B(z)| = 1$ on $\mathbb{T}$).

Theorem 1.2. *Given (small) $\delta > 0$, there exist finite Blaschke products $f_1, f_2 \in H^\infty$ with simple zeroes, satisfying*

$$2 \geqslant |f_1(z)| + |f_2(z)| \geqslant \delta > 0 \qquad \forall z \in \mathbb{D},$$

and such, that for any $g_1, g_2 \in H^\infty$ satisfying $f_1 g_1 + f_2 g_2 \equiv 1$, the inequality $\|g_1\|_\infty \geqslant C\delta^{-2}\log(-\log\delta)$ (where C is an absolute constant) holds.

As it will be shown later in Section 3, this result implies Theorem 1.1.

2. Interpolation and estimates of the solution of the (generalized) Corona Problem

In this section we present a well known connection of the (generalized) Corona Problem with an interpolation problem. Let us consider the case $n = 2$. Suppose we are given functions $f_1, f_2, f \in H^\infty$ and we want to find functions $g_1, g_2 \in H^\infty$ satisfying

$$f_1 g_1 + f_2 g_2 = f. \tag{2.1}$$

Suppose, that the function f_2 is a Blaschke product with simple zeroes, and let σ be the set of its zeroes. Then any function g_1 from (2.1) has to be a solution of the following interpolation problem:

$$g_1(\lambda) = f(\lambda)/f_1(\lambda) \qquad \forall \lambda \in \sigma := f_1^{-1}(0). \tag{2.2}$$

Moreover, any solution g_1 of the interpolation problem (2.2) gives rise to a solution of the Corona Problem (2.1) (just put $g_2 = (f - f_1 g_1)/f_2$).

The general form of the solution g_1 of the interpolation problem (2.2) is $g_1 = g_1^0 + f_2\varphi$, $\varphi \in H^\infty$, where g_1^0 is some fixed solution. Consider a bounded linear functional L on H^1,

$$Lh := \frac{1}{2\pi i}\int_{\mathbb{T}} g_1^0(z)h(z)/f_2(z)dz, \qquad h\in H^1.$$

By the Hahn–Banach Theorem the functional L admits an extension to L^1 with the same norm. Since all bounded linear functionals $\widetilde{L}$ on L^1 coinciding with L on H^1 can be described by the formula

$$\widetilde{L}h := \frac{1}{2\pi i}\int_{\mathbb{T}} (g_1^0(z)/f_2(z)+\varphi(z))h(z)dz, \qquad \varphi\in H^\infty, \qquad h\in L^1,$$

we have

$$\begin{aligned}\|L\| = \inf\{\|\widetilde{L}\| \,:\, \widetilde{L}\in (L^1)^*,\, \widetilde{L}\,\big|_{H^1}= L\} &= \inf_{\varphi\in H^\infty}\|(g_1^0/f_2+\varphi)\|_\infty\\ &= \inf_{\varphi\in H^\infty}\|(g_1^0+f_2\varphi)\|_\infty\end{aligned}$$

So, the smallest possible L^∞-norm of the solution g_1 is

$$\|L\| = \sup\left\{\frac{1}{2\pi i}\int_{\mathbb{T}} g_1(z)h(z)/f_2(z)dz \;:\; h\in H^1, \|h\|_1\leqslant 1\right\}.$$

(clearly, the integral does not depend on the choice of g_1). The integral inside the supremum can be computed using residues

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}} g_1(z)h(z)/f_2(z)dz &= \sum_{\lambda\in\sigma} g_1(\lambda)h(\lambda)\operatorname{res}_\lambda(1/f_2)\\ &= \sum_{\lambda\in\sigma}\frac{f(\lambda)}{f_1(\lambda)}h(\lambda)\operatorname{res}_\lambda(1/f_2).\end{aligned}$$

In what follows f_2 will be a finite Blaschke product, so we will not have to worry about convergence of the sum.

3. From estimates to ideals

In this section we will show how to get Theorem 1.1 from Theorem 1.2.

Pick a sequence of $\delta_k>0$, $\delta_k\searrow 0$, say $\delta_k = 2^{-k}$.

By theorem 1.2 there exist finite Blaschke products f_1^k, f_2^k with simple zeroes (k is an index, not an exponent here), satisfying

$$|f_1^k(z)|+|f_2^k(z)|\geqslant \delta_k$$

and such that any $g_1^k\in H^\infty$ satisfying $f_1^kg_1^k+f_2^kg_2^k\equiv 1$ for $g_2^k\in H^\infty$ is estimated from below $\|g_1^k\|_\infty\geqslant C\delta^{-2}\log(-\log\delta)$. As it was discussed above in Section 2 any such g_1 solves the interpolation problem on zeroes of f_2^k:

$$g_1^k(\lambda) = 1/f_1^k(\lambda), \qquad \forall\lambda\in\sigma_k := \{z\in\mathbb{C} : f_2^k(z)=0\}.$$

Therefore, any g_1^k solving the interpolation problem

$$g_1^k(\lambda) = (\delta_k)^2/f_1^k(\lambda) =: a_\lambda^k, \qquad \forall\lambda\in\sigma_k$$

satisfies the estimate $\|g_1^k\|_\infty\geqslant C\log(-\log\delta)$.

Let us remind the reader, that we interpolate in finitely many points, so, if we perturb the interpolation data a^k_λ a little, the solution of the interpolation problem still has big norm. Namely, there exists $\varepsilon_k > 0$ such that for any $\tilde a^k_\lambda$ satisfying $|\tilde a^k_\lambda - a^k_\lambda| < \varepsilon_k$ the solution g_k of the interpolation problem

$$g_k(\lambda) = \tilde a^k_\lambda, \qquad \forall \lambda \in \sigma_k \tag{3.1}$$

admits the estimate $\|g_k\|_\infty \geqslant (C/2)\log(-\log\delta_k)$.

This follows, for example, from the fact, that Lagrange's interpolating polynomial depends continuously on the interpolation data.

Now it is convenient to move everything to the upper half-plane, using the conformal mapping from the unit disc to the upper half-plane. So, from now to the end of this section we assume that all functions are bounded analytic functions on the upper half-plane.

Define outer functions h_k by

$$h_k(z) = \exp\left\{\frac{1}{\pi i}\int_{I_k}\frac{\delta_k}{t-z}dt\right\},$$

where I_k are some appropriate intervals. The functions h_k are outer functions such that

$$|h_k(t)| = \begin{cases} \delta_k, & t \in I_k, \\ 1, & t \in \mathbb{R}\setminus I_k, \end{cases}$$

and such that $h_k(\infty) = 1$.

We assume that intervals I_k are sufficiently large, and that the sets σ_k (zeroes of f^k_2) are close to centers of I_k, so the values of h_k on the set σ_k are close to δ_k. Namely, we assume that

$$|\, a^k_\lambda - h_k(\lambda)^2/f^k_1(\lambda)| < \varepsilon_k/2 \qquad \forall\lambda\in\sigma_k,$$

where, let us recall, $a^k_\lambda := (\delta_k)^2/f^k_1(\lambda)$.

Let us recall that the finite Blaschke products f^k_1, f^k_2 satisfy the estimate

$$|f^k_1(z)| + |f^k_2(z)| \geqslant \delta_k$$

Since for a finite Blaschke product B on the upper half-plane we have $\lim_{|z|\to\infty}|B(z)| = 1$, by picking sufficiently large intervals I_k we can always assume that

$$|f^k_1(z)| + |f^k_2(z)| \geqslant |h_k(z)|/2 \qquad \forall z\in\mathbb{C}_+.$$

Shifting the arguments of f^k_1, f^k_2 and h_k by $\tau_k\in\mathbb{R}$, $\tau_k\to\infty$, we can always assume that the products

$$f_1(z) = \prod_k f^k_1(z-\tau_k), \qquad f_2 := \prod_k f^k_2(z-\tau_k), \qquad h := \prod_k h_k(z-\tau_k)$$

converge (here we assume that the Blaschke products $f^k_{1,2}$ are also normalized by the condition $f^k_{1,2}(\infty) = 1$). Moreover we can always take τ_k to be sufficiently large, so the all terms in products are almost independent, so

$$|f_1(z)| + |f_2(z)| \geqslant |h(z)|/4 \qquad \forall z\in\mathbb{C}_+.$$

and for all k

(3.2) $\quad | a^k_\lambda - h(\lambda)^2/f_1(\lambda)| < \varepsilon_k \qquad \forall\lambda$ such, that $\lambda - \tau_k \in \sigma_k$.

Suppose, now functions $g_1, g_2 \in H^\infty$ satisfy $f_1g_1 + f_2g_2 \equiv h^2$. Then g_1 solves the interpolation problems

$$f_1(\lambda) = h(\lambda)/f_1(\lambda) =: \tilde{a}^k_\lambda, \qquad \lambda - \tau_k \in \sigma_k,$$

$k = 1, 2, 3, \ldots$

Equation (3.2) means that $|\tilde{a}^k_\lambda - a^k_\lambda| < \varepsilon_k$, and so (see (3.1)) one can estimate $\|g_1\|_\infty \geqslant (C/2)\log(-\log\delta_k)$. Therefore (since $\delta_k \to 0$) $g_1 \notin H^\infty$, and we get the contradiction. $\square$

4. Tolokonnikov's example

In this section we are going to present Tolokonnikov's example [8] with the estimate C/δ^2. This example gives us a better understanding of what is going on in the next section where we prove Theorem 1.2.

Pick a small $\delta > 0$. Consider a Blaschke factor

$$B(z) = b_\delta(z) = \frac{z - \delta}{1 - \delta z}$$

Put $f_1(z) := \delta z$ and $f_2(z) := B(z^n)$ where n is sufficiently large. Clearly $|f_1| + |f_2| \geqslant \delta/2$ if n is large enough. We want to show that any solution of the Corona Problem has large norm.

Let $g_1, g_2 \in H^\infty$ satisfy $f_1g_1 + f_2g_2 \equiv 1$.

As we discussed above in Section 2 we need to estimate from below

$$\begin{aligned} L(h) := \frac{1}{2\pi i}\int_{\mathbb{T}} g_1(z)h(z)/f_2(z)dz &= \sum_{\lambda\in\sigma(f_2)} g_1(\lambda)h(\lambda)\,\mathrm{res}_\lambda(1/f_2) \\ &= \sum_{\lambda\in\sigma(f_2)} \frac{1}{f_1(\lambda)} h(\lambda)\,\mathrm{res}_\lambda(1/f_2), \end{aligned}$$

for $h \in H^1$, $\|h\|_1 = 1$.

The choice of h is simple: we just put $h \equiv 1$.

Let $\sigma(f)$ denote the zero set of the function $f \in H^\infty$. Then

$$L(1) = \sum_{\lambda\in\sigma(f_2)} \frac{1}{\delta\lambda}\,\mathrm{res}_\lambda(1/f_2).$$

Since f_2 has simple zeroes, $\mathrm{res}_\lambda(1/f_2) = 1/f_2'(\lambda)$.

The further computations are just simple and straightforward calculus exercises, but let us give some rather detailed explanation to highlight main idea.

Note that zeroes of f_2 are exactly the solutions of $\lambda^n = \delta$, so for such λ

$$f_2'(\lambda) = B'(\lambda^n)n\lambda^{n-1} = B'(\delta)n\lambda^{n-1} = \frac{1}{1-\delta^2}n\lambda^{n-1}$$

Therefore we have

$$L(1) = \sum_{\lambda:\lambda^n=\delta} \frac{1-\delta^2}{\delta n \lambda^n} = \sum_{\lambda:\lambda^n=\delta} \frac{1-\delta^2}{\delta^2 n} = \frac{1-\delta^2}{\delta^2},$$

and so $\|g_1\|_\infty \geqslant (1-\delta^2)/\delta^2$.

Let us discuss now what have happened, how did an extra δ appear in the denominator. If we analyze the construction we can see, that we took two functions $\varphi_1 \equiv \delta$ and $\varphi_2 = B$, satisfying the corona condition $|\varphi_1|+|\varphi_2| \geqslant \delta$. For this pair there clearly is a solution of norm $1/\delta$ ($g_1 \equiv 1/\delta$), and we wan to get an extra δ in the denominator. And to do this we composed the functions wit the map $z \mapsto z^n$.

However, by simply considering $f_1(z) = \varphi_1(z^n)$, $f_2 = \varphi_2(z^n)$ we do not gain anything: $g_1(z^n)$, $g_2(z^n)$ will be a solution of the corresponding Corona problem. So we used a little twist: we put $f_2(z) := \varphi_2(z^n)$ but $f_1(z) := z\varphi_1(z^n)$. If $|\varphi_2(z)| \geqslant \delta/2$ in a small neighborhood of the origin, extra z does not spoil the corona condition significantly: since the zeroes of φ_2 tend to the boundary as $n \to \infty$, we can guarantee that $|f_1|+|f_2| \geqslant \delta/2$ if n is large enough.

The extra z guarantee that we don't have cancellation when we compute $L(h)$ using residues. And when we compute residues we need to put $\varphi_2'(z) = \varphi'(z^n)nz^{n-1}$ in the denominator. If we get into account an extra z we get z^n in the denominator (n is compensated, because for each zero of φ_2 we have n zeroes of f_2). But $z^n = \delta$ and that is where an extra δ appears!

There is another way to see why a better estimate appears. Let us compare

$$\int_\gamma \frac{h(z)}{\varphi_1(z))\varphi_2(z)} dz \tag{4.1}$$

and

$$\int_\Gamma \frac{h(z^n)}{f_1(z)} \cdot \frac{1}{f_2(z)} dz,$$

where γ and Γ are the contours surrounding zeroes of φ_2 and f_2 respectively (zeroes of φ_2 and f_1 are outside of the corresponding contours). The second integral can be rewritten as

$$\int_\Gamma \frac{h(z^n)}{\varphi_1(z^n)} \cdot \frac{1}{\varphi_2(z^n)} \cdot \frac{dz}{z} = \int_\gamma \frac{h(w)}{\varphi_1(w)} \cdot \frac{1}{\varphi_2(w)} \cdot \frac{dw}{w}$$

($w = z^n$, $dw/w = ndz/z$; n cancels out because the mapping $z \mapsto z^n$ is n to 1). So, in comparison with (4.1) we got an extra z in the denominator, so we can hope to get something bigger.

We can get the same result by replacing in φ_1 by $z\varphi_1(z)$, but this in most cases will significantly spoil the Corona condition. And as we discussed above, for $f_1 = z\varphi_1(z^n)$ under very minimal assumptions on φ_2 the Corona Condition still holds.

So, that is the main trick. In the next section (Section 5) we apply it twice to get an estimate better than $1/\delta^2$.

To conclude the section, let us note, that there is a simpler way to check that Tolokonnikov's construction gives the estimate C/δ^2: change the role of f_1 and f_2 and compute the integral using the (only) residue of f_1 at 0. An extra δ appears because $|f_1(z)| = \delta$ for $z \in \mathbb{T}$, not to 1 (i. e. f_1 is not an inner function, but δ times inner). However, this simpler computation gives no insight whatsoever into the construction in the next section.

5. Estimates in the Corona Theorem: proof of Theorem 1.2

Let $B = b_{-\delta^2}$ be the Blaschke factor

$$B(z) := \frac{z + \delta^2}{1 + \delta^2 z}.$$

Clearly $|B(t)| \geqslant \delta^2$ for real $t \in [0, 1]$.

For a function φ let $\sigma(\varphi)$ denote the set of its zeroes. Define $\varphi(z) = B(z^{2n})$, where n is a large integer to be chosen later, and let

$$\varphi_1(z) := \prod_{\substack{\lambda\in\sigma(\varphi),\\ \operatorname{Im}\lambda>0}} \frac{z-\lambda}{1-\overline{\lambda}z}, \qquad \varphi_2(z) := \prod_{\substack{\lambda\in\sigma(\varphi),\\ \operatorname{Im}\lambda<0}} \frac{z-\lambda}{1-\overline{\lambda}z}.$$

Clearly, $\varphi = \varphi_1\varphi_2$. Indeed, it is trivial, that $\varphi = \xi\varphi_1\varphi_2$, where ξ is a unimodular ($|\xi| = 1$) constant, and since $\varphi_1(0)\varphi_2(0) > 0$, we can conclude that $\xi = 1$.

Note, that the zero set $\sigma(\varphi)$ consists of $2n$ points $\delta^{1/n}\xi_k$, where ξ_k are $2n$th roots of -1, $\xi_k^{2n} = -1$, see Figure 1.

Let $a = \delta^{1/n}$, and let $\omega(z) = \frac{z-a}{1-az}$. Define

$$\psi_j(z) = \varphi_j \circ \omega, \qquad j = 1, 2.$$

Clearly $\sigma(\psi_j) = \omega^{-1}(\sigma(\varphi_{1,2}))$. Note, that the inverse map ω^{-1} is given by the formula $\omega^{-1}(z) = \frac{z+a}{1+az}$

Pick a sufficiently large m, and define $f_1(z) = z\psi_1(z^m)$, $f_2(z) = \psi_2(z^m)$.

We claim that for sufficiently large m the functions f_1, f_2 satisfy the Corona Condition $|f_1| + |f_2| \geqslant \delta/2$.

Indeed, since $|B(t)| \geqslant \delta^2$ for $t \in [0, 1]$, it follows that $|\varphi(t)| \geqslant \delta^2$ for $t \in [-1, 1]$. The symmetry between φ_1 and φ_2 implies that $|\varphi_j(t)| \geqslant \delta$, $j = 1, 2$ for $t \in [-1, 1]$. The maximum modulus principle implies that $|\varphi_2(z)| \geqslant \delta$ for $z \in \mathbb{D}$, $\operatorname{Im} z \geqslant 0$ and that $|\varphi_1(z)| \geqslant \delta$ for $z \in \mathbb{D}$, $\operatorname{Im} z \leqslant 0$. Therefore

$$|\varphi_1(z)| + |\varphi_2(z)| \geqslant \delta \qquad \forall z \in D,$$

and the same inequality clearly holds for ψ_1, ψ_2.

Since ω maps the interval $[-1, 1]$ onto itself, $|\psi_j(0)| \geqslant \delta$, $j = 1, 2$, and therefore in a small ε-neighborhood of the origin $|\psi_2(z)| \geqslant \delta/2$.

If m is sufficiently large, so $\varepsilon^{1/m} \geqslant 1/2$, the functions f_1, f_2 clearly satisfy the Corona Condition $|f_1| + |f_2| \geqslant \delta/2$.

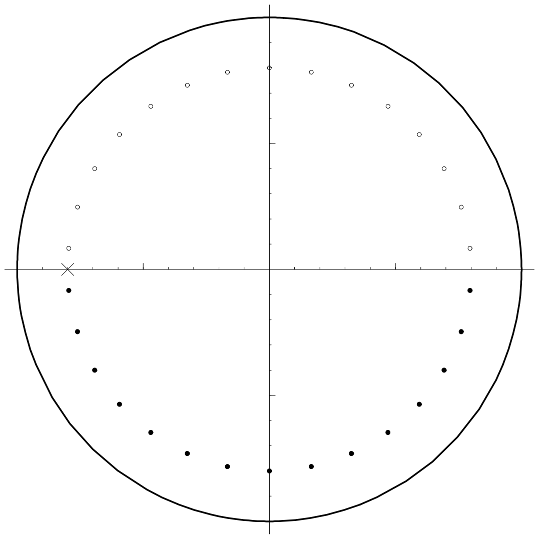

FIGURE 1. Zeroes of φ_1 (hollow points) and of φ_2 (solid points). Point $-a$ is marked by the $\times$.

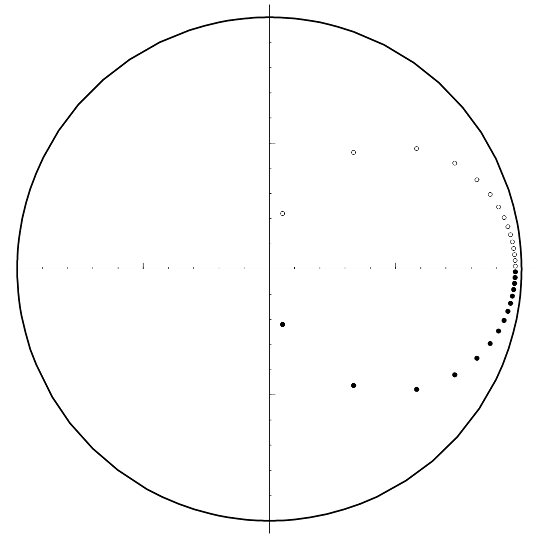

FIGURE 2. Zeroes of ψ_1 (hollow points) and of ψ_2 (solid points).

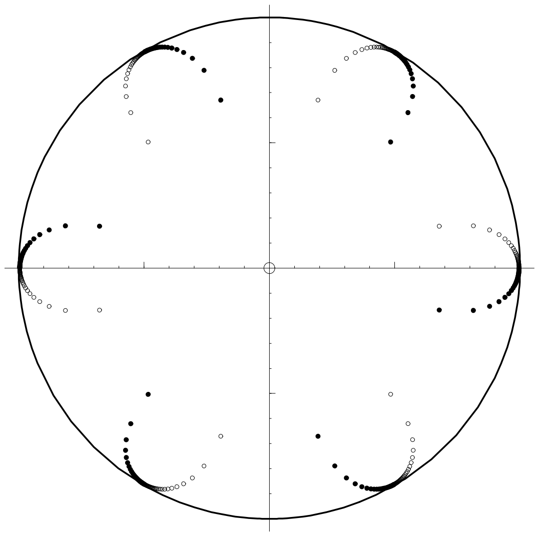

FIGURE 3. Zeroes of f_1 (hollow points) and of f_2 (solid points). Note, that the origin is a zero of f_1.

We want to show that any solution of the Corona problem has large norm. As it was discussed above in Section 2, we need to pick $h \in H^1$, $\|h\|_1 = 1$ and estimate below

$$L(h) := \frac{1}{2\pi i}\int_{\mathbb{T}} g_1(z)h(z)/f_2(z)dz = \sum_{\lambda\in\sigma(f_2)} g_1(\lambda)h(\lambda)\operatorname{res}_\lambda(1/f_2)$$
$$= \sum_{\lambda\in\sigma(f_2)} \frac{1}{f_1(\lambda)}h(\lambda)\operatorname{res}_\lambda(1/f_2).$$

We put $h(z) \equiv 1$, and estimate $L(1)$. Since all zeroes of f_2 are simple,

$$\operatorname{res}_\lambda(1/f_2) = 1/f_2'(\lambda), \qquad \lambda \in \sigma(f_2).$$

Recalling the definition of $f_{1,2}$ we get

$$L(1) = \sum_{\lambda\in\sigma(f_2)} \frac{1}{\lambda\psi_1(\lambda^m)} \cdot \frac{1}{\psi_2'(\lambda^m)\cdot m\lambda^{m-1}} = \sum_{\lambda\in\sigma(f_2)} \frac{1}{m\lambda^m\psi_1(\lambda^m)\psi_2'(\lambda^m)}.$$

Map $\lambda \to \lambda^m$ maps zeroes of f_2 onto zeroes of ψ_2, and to each zero of f_2 correspond exactly m zeroes of ψ_2, so we can rewrite

$$L(1) = \sum_{z\in\sigma(\psi_2)} \frac{1}{z\psi_1(z)\psi_2'(z)} = \sum_{z\in\sigma(\psi_2)} \frac{1}{z\psi'(z)};$$

here we use identity $\psi'(z) = \psi_1(z)\psi_2'(z) + \psi_1'(z)\psi_2(z) = \psi_1(z)\psi_2'(z)$ if $\psi_2(z) = 0$. Note an extra z in the denominator: because of it we will get large $L(1)$.

Recalling that $\psi = \varphi \circ \omega$, and using the chain rule, we get

$$L(1) = \sum_{z\in\sigma(\psi_2)} \frac{1}{z\varphi'(\omega(z))\omega'(z)}.$$

Denote $\lambda = \omega(z)$, and let α is the inverse map for ω, $\alpha(\lambda) = \omega^{-1}(\lambda) = (\lambda + a)/(1 + a\lambda)$. Then we can rewrite (using the identity $\alpha'(\lambda) = (1 - a^2)/(1 + a\lambda)^2$)

$$L(1) = \sum_{\lambda\in\sigma(\varphi_2)} \frac{\alpha'(\lambda)}{\alpha(\lambda)\varphi'(\lambda)} = \sum_{\lambda\in\sigma(\varphi_2)} \frac{1 - a^2}{(\lambda + a)(1 + a\lambda)\varphi'(\lambda)}.$$

Note that the zero set $\sigma(\varphi_2)$ consists of all the solutions of the equation $z^{2n} = -\delta^2$ with $\operatorname{Re} z < 0$. Recalling that $\varphi = B(z^{2n})$ we get

$$\varphi'(z) = B'(-\delta^2)2nz^{2n-1} = 2n\delta^2/[z\cdot(1 - \delta^4)], \qquad z \in \sigma(\varphi_2).$$

So, to estimate $L(1)$ we need to estimate the sum

$$\begin{aligned} L(1) &= \frac{(1-\delta^4)}{2\delta^2} \sum_{\lambda\in\sigma(\varphi_2)} \frac{(1 - a^2)\lambda}{(\lambda + a)(1 + a\lambda)n} \\ &= \frac{(1-\delta^4)}{2\delta^2}\Bigg(\sum_{\substack{\lambda\in\sigma(\varphi_2)\\ |a+\lambda|<(1-a^2)/10}} \dots \quad + \sum_{\substack{\lambda\in\sigma(\varphi_2)\\ |a+\lambda|\geqslant(1-a^2)/10}} \dots \Bigg) \\ &= \frac{(1-\delta^4)}{2\delta^2}\big(\Sigma_1 + \Sigma_2\big). \end{aligned}$$

Note, that the distance between poits $\lambda \in \sigma(\varphi_2)$ is about π/n, and $1 - a^2 = 1 - \delta^{2/n} \sim (2/n)\log\delta^{-1}$, so the sum Σ_1 contains about $(2/\pi)\log\delta^{-1}$ terms. Let us recall that $a = \delta^{1/n}$, $|\lambda| = \delta^{1/n}$, and n is in our power and we can make it as big as we want. So we can assume that a is close to one, and therefore in the sum Σ_1

$$\Big|\frac{(1 - a^2)\lambda}{1 + a\lambda} + 1\Big| < 1/5.$$

Indeed,

$$\frac{(1 - a^2)\lambda}{1 + a\lambda} = \frac{(1 + a\lambda - a\lambda - a^2)\lambda}{1 + a\lambda} = \lambda - \frac{(\lambda + a)a\lambda}{1 + a\lambda};$$

since $a = |\lambda| = \delta^{1/n}$ and $|a + \lambda| \leqslant (1 - |a|^2)/10$ in Σ_1, then increasing n we can make λ as close to -1 as we want. Since $|1 + a\lambda| \geqslant 1 - a^2$, the second term can be estimated by $|a\lambda|/10$.

Moreover, since all λ in Σ_1 lie almost on a straight segment (for sufficiently large n), we can assume that all the terms in Σ_1 lie in a sector with vertex at the origin with the angle of, say $\pi/2$. Thus, to estimate Σ_1 from below we need to estimate

$$\sum_{\substack{\lambda\in\sigma(\varphi_2)\\ |a+\lambda|<(1-a^2)/10}} \frac{1}{n\cdot|\lambda+a|}\,.$$

The latter sum can be estimated from below by the integral (up to some absolute multiplicative constant)

$$\int_{1/n}^{(1-a^2)/10}\frac{dx}{x} = \log[(1-a^2)/10] - \log(1/n) = \log(n(1-a^2)) - \log 10$$

(consider Riemann sums with size of partition about π/n; recall that the distance between points of $\sigma(\varphi_2)$ is about π/n).

Taking into account that

$$\lim_{n\to\infty} n\cdot(1-a^2) = \lim_{n\to\infty} n\cdot(1-\delta^{2/n}) = \lim_{n\to\infty} n\cdot(1-e^{(2/n)\log\delta}) = -2\log\delta$$

we get the estimate $|\Sigma_1| \geqslant c\log(-\log\delta)$ for small δ.

The second sum Σ_2 is estimated from above by the integral (again up to an absolute multiplicative constant)

$$|\Sigma_2| \leqslant \sum_{\substack{\lambda\in\sigma(\varphi_2)\\ |a+\lambda|\geqslant(1-a^2)/10}} \left|\frac{(1-a^2)\lambda}{(\lambda+a)(1+a\lambda)n}\right| \leqslant C\int_{(1-a^2)/10}^{2}\frac{1-a^2}{x^2}dx = A < \infty$$

(again consider Riemann sums).

Combining all estimates together we obtain that

$$|L(1)| \geqslant \frac{C}{\delta^2}\cdot\log(-\log\delta)$$

for small δ. Therefore $\|g_1\|_\infty \geqslant C\delta^{-2}\log(-\log\delta)$ and we are done. □

6. Concluding remarks.

There is still a gap between lower bound $\delta^{-2}\log\log\delta^{-1}$ obtained in this paper and the best known upper bound $\delta^{-2}\log\delta^{-1}$ [6, 9]. The author does not know which one of the estimates (if either) is sharp.

One of the important (at least for the author) "corollaries" of the estimates is that they show that it is extremely unlikely to get an "operator theory" proof of the Corona Theorem. While it looked plausible that one could get an estimate δ^{-2} by operator means, it is hard to imagine how one can get an estimate between $\delta^{-2}\log\log\delta^{-1}$ and $\delta^{-2}\log\delta^{-1}$ in that fashion.

References

[1] J. Bourgain, *On finitely generated closed ideals in $H^\infty(D)$*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 163–174;

[2] U. Cegrell, *A generalization of the corona theorem in the unit disc*, Math. Z. **203** (1990), no. 2, 255–261.

[3] U. Cegrell, *Generalizations of the corona theorem in the unit disc*, (English. English summary) Proc. Roy. Irish Acad. Sect. A, **94** (1994), no. 1, 25–30.
[4] J. W. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
[5] V. P. Havin, S. V. Hruščëv, and N. K. Nikolskiĭ (eds.), *Linear and complex analysis problem book*, Springer-Verlag, Berlin, 1984, 199 research problems.
[6] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Springer–Verlag, 1986, 491 pp.
[7] K. V. R. Rao, *On a generalized corona problem*, J. Analyse Math, **18**(1967), 277–278.
[8] V. A. Tolokonnikov, *Estimates in the Carleson corona theorem, ideals of the algebra H^∞, a problem of Sz.-Nagy. (Russian. English summary)*, Investigations on linear operators and the theory of functions, XI. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 178–198, 267.
[9] A. Uchiyama, *Corona theorems for countably many functions and estimates for their solutions*, preprint, UCLA, 1980.
[10] T. Wolff, *A refinement of the corona theorem* In: V. P. Havin, S. V. Hruščëv, and N. K. Nikolskiĭ (eds.), *Linear and complex analysis problem book*, Springer-Verlag, Berlin, 1984, 199 research problems.

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