THE WEIGHTED NORM INEQUALITIES FOR HILBERT TRANSFORM ARE NOW TRIVIAL.

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The goal of this note is to give a very simple proof of the famous Hunt–Muckenhoupt–Wheeden Theorem [1]. We are going to prove that the Hilbert transform T (or, equivalently the Riesz projection P_+) is bounded in the weighted space $L^2(w) = L^2(\mathbb{T}, w)$ ($\mathbb{T}$ denotes the unit circle on the complex plane) if and only if the weight w satisfies the so-called Muckenhoupt (A_2) condition,

$$\sup_I \Big(\frac{1}{|I|}\int_I w\Big)\Big(\frac{1}{|I|}\int_I w^{-1}\Big) < \infty. \tag{A_2}$$

where the *supremum* is taken over all subarcs I of $\mathbb{T}$.

Let us recall that for a trigonometric polynomial $f = \sum_{-N}^{N} \hat f(n) z^n$, $|z| = 1$ the Riesz projection P_+ is defined by $P_+ f = \sum_{n\ge 0} \hat f(n) z^n$, and the Hilbert transform T is defined as $T = -iP_+ + iP_-$, where $P_- = I - P_+$.

The above result is very well known. The fact that the Muckenhoupt condition is necessary is very simple, so we will prove only the hard part, the sufficiency.

The proof we are presenting looks like a very clever artificial trick. It looks like it is impossible to understand how one can find that kind of tricks. However it is not the case, and there are some very deep ideas behind this elementary proof. Unfortunately, we can not explain the ideas in this short note, and if the reader wants to understand them, he should read our paper "The hunt for a Bellman function ... " that will appear in *St. Petersburg Mathematical Journal.*

We will use another form of the Muckenhoupt condition. Let us fix some notation first. Given a function f (say in L^1) on the unit circle $\mathbb{T}$, let $f(z)$, $z \in \mathbb{D}$ denote its harmonic extension at point z. In this notation $f^2(z)$ and $f(z)^2$ denote different things: the first is the harmonic extension of $f^2 \,\big|\, \mathbb{T}$ at the point $z \in \mathbb{D}$, and the latter is the square of the harmonic extension of f.

It is well known that the Muckenhoupt condition (A_2) is equivalent to

$$\sup_{z\in\mathbb{D}} w(z)\, w^{-1}(z) = A < \infty, \tag{invA_2}$$

which we will call the invariant (A_2) condition (because it is invariant under Möbius transformations of the circle). The invariant (A_2) condition clearly implies (A_2); the proof that they are equivalent is simple and well known. Or, if the reader does not like or does not know this proof, there is a simple proof that the condition (invA_2) is necessary for the Hilbert transform to be bounded, cf [2].

We can rewrite the condition (invA_2) as

$$\sup_{z\in\mathbb{D}} \big\{ [w(z)/\exp\{(\log w)(z)\}] \cdot \big[w^{-1}(z)/\exp\big\{(\log w^{-1})(z)\big\}\big] \big\} < \infty$$

(multiplying and dividing by the same number). By Jensen inequality expressions in brackets are at least 1, so the condition splits into the following two

$$\sup_{z\in\mathbb{D}} [w(z)/\exp\{(\log w)(z)\}] = A < \infty\,, \qquad (\mathrm{inv}A_\infty)$$

and the same condition for w^{-1}. By the analogy with the usual (A_∞) condition we will call the condition above the invariant (Möbius invariant) (A_∞) condition. The constant A in $(\mathrm{inv}A_\infty)$ is called the $(\mathrm{inv}A_\infty)$ norm of w. Clearly $w \in (\mathrm{inv}A_2) \iff w \in (\mathrm{inv}A_\infty)$ & $w^{-1} \in (\mathrm{inv}A_\infty)$.

The fanciest technical "tool" we are using is Green's formula which says that for any C^2 up to the boundary function F on $\mathbb{D}$,

$$\frac{1}{2\pi}\int_{\mathbb{T}} F(z)|dz| - F(0) = \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta F(z) \log\frac{1}{|z|}dxdy\,;$$

here Δ denotes the Laplacian.

Suppose we proved the following "imbedding theorem":

Theorem 1. *Let w satisfy the $(invA_\infty)$ condition. Then for any $f \in L^2(w^{-1})$*

$$\iint_{\mathbb{D}} |\nabla f(z)|^2 w(z)^{-1} \log\frac{1}{|z|}dxdy \le C\frac{1}{2\pi}\int_{\mathbb{T}} |f(z)|^2 w(z)^{-1}|dz| = C\|f\|^2_{L^2(w^{-1})},$$

where the constant C depends only on the $(invA_\infty)$ norm of w.

Then the Hunt–Muckenhoupt–Wheeden theorem follows almost immediately.

Proof of the Hunt–Muckenhoupt–Wheeden theorem. Consider operators $\partial = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)$ and $\overline{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$. It is easy to check that $|\nabla f|^2 = 2(|\partial f|^2 + |\overline{\partial} f|^2)$ and $\Delta = 4\partial\overline{\partial}$.

We are going to show that the operator $P_- = I - P_+$ is bounded in $L^2(w)$; that immediately implies that P_+ and the Hilbert transform T are bounded as well. Note that $(P_-f)(0) = 0$, and that $\partial P_- f = 0$.

By duality

$$\|P_-f\|_{L^2(w)} = \sup\Big\{\Big|\frac{1}{2\pi}\int_{\mathbb{T}}(P_-f)(z)g(z)|dz|\Big| \ :\ g\in L^2(w^{-1}),\ \|g\|_{L^2(w^{-1})}\le 1\Big\}\,.$$

Since $P_-f(0) = 0$, Green's formula [1] and the identity $\Delta = 4\partial\overline{\partial}$ imply

$$\begin{aligned}
\Big|\frac{1}{2\pi}\int_{\mathbb{T}}(P_-f)(z)g(z)|dz|\Big| &= \Big|\frac{1}{2\pi}\iint_{\mathbb{D}} \Delta\Big((P_-f)(z)g(z)\Big)\log\frac{1}{|z|}dxdy\Big| = \\
&= \Big|\frac{2}{\pi}\iint_{\mathbb{D}} \overline{\partial} f(z)\cdot\partial g(z)\cdot\log\frac{1}{|z|}dxdy\Big| \le \Big(\frac{2}{\pi}\iint_{\mathbb{D}}|\nabla g(z)|^2 w(z)^{-1}\log\frac{1}{|z|}dxdy\Big)^{1/2}\times \\
&\times\Big(\frac{2}{\pi}\iint_{\mathbb{D}}|\nabla f(z)|^2 w(z)w^{-1}(z)[w^{-1}(z)]^{-1}\log\frac{1}{|z|}dxdy\Big)^{1/2} \le \\
&\le C\|g\|_{L^2(w^{-1})}\Big(\frac{2A}{\pi}\iint_{\mathbb{D}}|\nabla f(z)|^2[w^{-1}(z)]^{-1}\log\frac{1}{|z|}dxdy\Big)^{1/2} \le C'\|g\|_{L^2(w^{-1})}\|f\|_{L^2(w)}\,,
\end{aligned}$$

[1] We can always assume that functions f, g, w, w^{-1} are smooth enough on $\mathbb{T}$, so we do not have to worry about boundary values. Then, since we will obtain an estimate independent of the smoothness we can use standard approximation argument to get the general result.

(the last inequality follows from Theorem 1 for w^{-1}) and the result is proved modulo the imbedding theorem (Theorem 1) $\square$

Proof of Theorem 1. Clearly, it is enough to prove the theorem only for real valued functions f. Again it is enough to prove it only for smooth enough functions f, w, w^{-1}.

Consider the function Φ of three real variables f, w and ω, defined by

$$\Phi(f,w,\omega) = \frac{f^2}{w}\Big(1+\frac{\delta}{(we^{-\omega})^\alpha}\Big),$$

where $\alpha > 1$ and $0 < \delta \le (\alpha-1)/4\alpha$. It is easy to check and will be shown later that

$$d^2\Phi \ge \delta'(we^{-\omega})^{-\alpha}\frac{(df)^2}{w},$$

where $\delta' = \delta'(\delta,\alpha) > 0$.

If we have a C^2-smooth function F of n variables x_1, x_2, ... ,x_n and n harmonic functions u_1, u_2, ... ,u_n , then

$$\Delta F(u_1,u_2,\dots,u_n) = \sum_{j,k=1}^n \frac{\partial^2 F}{\partial x_j \partial x_k}\langle \nabla u_j, \nabla u_k\rangle .$$

Applying this to $F = \Phi$ and harmonic functions $f = f(z)$, $w = w(z)$, $\omega = (\log w)(z)$, one can conclude that

$$\Delta\Phi(f(z),w(z),\omega(z)) \ge \delta'\big[w(z)e^{-(\log w)(z)}\big]^{-\alpha}\frac{|\nabla f(z)|^2}{w(z)} \ge \delta' A^{-\alpha}\frac{|\nabla f(z)|^2}{w(z)} .$$

where $A = \sup_{z\in D}\big[w(z)e^{-(\log w)(z)}\big]$ is the (invA_∞) norm of w. Since for every non-negative smooth enough function F on $\mathbb{D}$ we have by Green's formula

$$\frac{1}{2\pi}\iint_{\mathbb{D}} \Delta F(z)\log\frac{1}{|z|}dxdy = \frac{1}{2\pi}\int_{\mathbb{T}} F(z)|dz| - F(0) \le \frac{1}{2\pi}\int_{\mathbb{T}} F(z)|dz| ,$$

we get for $F(z) = \Phi(f(z),w(z),(\log w)(z))$

$$\frac{1}{2\pi}\iint_{\mathbb{D}} \delta' A^{-\alpha}\frac{|\nabla f(z)|^2}{w(z)}\log\frac{1}{|z|}dxdy \le \frac{1}{2\pi}\int_{\mathbb{T}} F(z)|dz| = \frac{1}{2\pi}\int_{\mathbb{T}}(1+\delta)\frac{f(z)^2}{w(z)}|dz|$$

which is equivalent to the imbedding theorem we wanted to prove with constant $C = A^\alpha\cdot\dfrac{1+\delta}{\delta'}$. $\square$

Appendix A. An elementary proof of Dechao Zheng's theorem.

As an additional bonus we are going to give an elementary proof of a result by Dechao Zheng [3] that if two weights v and w satisfy for some $\beta > 1$

$$\sup_{z\in\mathbb{D}} v^\beta(z)w^{-\beta}(z) < \infty ,$$

then the Hilbert transform is a bounded operator from $L^2(w)$ to $L^2(v)$, which means that

$$\int_{\mathbb{T}} |Tf|^2 v \le C\int_{\mathbb{T}} |f|^2 w .$$

All we need for that is to replace Theorem 1 by the following imbedding theorem.

Theorem 2. *If $v \geq 0$ and $v \in L^\beta$, $\beta > 1$, then for any $f \in L^2(v^{-1})$*

$$\iint_{\mathbb{D}} |\nabla f(z)|^2 v^\beta(z)^{-1/\beta} \log\frac{1}{|z|} dxdy \leq C\frac{1}{2\pi}\int_{\mathbb{T}} |f(z)|^2 v(z)^{-1}|dz| = C\|f\|^2_{L^2(v^{-1})}$$

with some absolute constant C.

Proof. Consider a function Φ of two real variables f and u,

$$\Phi(f,u) = \frac{f^2}{u^{1/\beta}}\,.$$

It can be easily verified that the Hessian of this function is strictly positively defined, and moreover

$$d^2\Phi \geq \delta\frac{(df)^2}{u^{1/\beta}}\,, \qquad \delta > 0\,.$$

Let now $f = f(z)$ and $u = v^\beta(z)$, $z \in \mathbb{D}$, so Φ becomes a function on $\mathbb{D}$. Then acting as in the proof of Theorem 1, one can get

$$\Delta\Phi\big(f(z), v^\beta(z)\big) \geq \delta\frac{|\nabla f|^2}{[v^\beta(z)]^{1/\beta}}\,.$$

Applying Green's formula (as in the proof of Theorem 1), we get the desired inequality. $\square$

Proof of Dechao Zheng's theorem. It is enough to show that the operator $P_- = I - P_+$ is a bounded operator from $L^2(w)$ to $L^2(v)$. Note that $(P_-f)(0) = 0$, and that $\partial P_- f = 0$.

Again we may assume that functions f, g, w, w^{-1} are smooth enough on $\mathbb{T}$, so we do not have to worry about boundary values.

By duality

$$\|P_-f\|_{L^2(v)} = \sup\left\{\left|\frac{1}{2\pi}\int_{\mathbb{T}} f(z)g(z)|dz|\right| \ :\ g \in L^2(v^{-1}),\ \|g\|_{L^2(v^{-1})} \leq 1\right\}.$$

Since $P_-f(0) = 0$, Green's formula and the identity $\Delta = 4\partial\overline{\partial}$ imply

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}} (P_-f)(z)g(z)|dz|\right| = \left|\frac{1}{2\pi}\iint_{\mathbb{D}} \Delta\Big((P_-f)(z)g(z)\Big)\log\frac{1}{|z|}dxdy\right| =$$

$$= \left|\frac{2}{\pi}\iint_{\mathbb{D}} \overline{\partial} f(z)\cdot\partial g(z)\cdot\log\frac{1}{|z|}dxdy\right| \leq \left(\frac{2}{\pi}\iint_{\mathbb{D}} |\nabla g(z)|^2[v^\beta(z)]^{-1/\beta}\log\frac{1}{|z|}dxdy\right)^{1/2} \times$$

$$\times\left(\frac{2}{\pi}\iint_{\mathbb{D}} |\nabla f(z)|^2[v^\beta(z)]^{1/\beta}[w^{-\beta}(z)]^{1/\beta}[w^{-\beta}(z)]^{-1/\beta}\log\frac{1}{|z|}dxdy\right)^{1/2} \leq$$

$$\leq C\|g\|_{L^2(v^{-1})}\left(\frac{2A}{\pi}\iint_{\mathbb{D}} |\nabla f(z)|^2[w^{-\beta}(z)]^{-1/\beta}\log\frac{1}{|z|}dxdy\right)^{1/2} \leq C'\|g\|_{L^2(v^{-1})}\|f\|_{L^2(w)}$$

(the last inequality is just Theorem 2 for $v = w^{-1}$), and the result is proved. $\square$

Appendix B. Computation of $d^2\Phi$.

The reader can check (using pen and paper, or feeding it into "Mathematica" or any other symbolic manipulation program) that for the function Φ from the proof of Theorem 1

$$d^2\Phi = \delta\frac{f^2}{w}\cdot \begin{pmatrix} \dfrac{df}{f}-\dfrac{dw}{w} \\ \dfrac{dw}{w}-d\omega \\ d\omega \end{pmatrix}^T \begin{pmatrix} \dfrac{2}{\delta}+2x^{-\alpha} & -2\alpha x^{-\alpha} & 0 \\ -2\alpha x^{-\alpha} & \alpha(\alpha-1)x^{-\alpha} & 0 \\ 0 & 0 & \alpha x^{-\alpha} \end{pmatrix} \begin{pmatrix} \dfrac{df}{f}-\dfrac{dw}{w} \\ \dfrac{dw}{w}-d\omega \\ d\omega \end{pmatrix},$$

where $x = we^{-\omega}$. If $\delta \le (\alpha-1)/4\alpha$ then

$$d^2\Phi \ge \delta\frac{f^2}{w}\cdot \begin{pmatrix} \dfrac{df}{f}-\dfrac{dw}{w} \\ \dfrac{dw}{w}-d\omega \\ d\omega \end{pmatrix}^T \begin{pmatrix} 2x^{-\alpha} & 0 & 0 \\ 0 & \dfrac{1}{2}\alpha(\alpha-1)x^{-\alpha} & 0 \\ 0 & 0 & \alpha x^{-\alpha} \end{pmatrix} \begin{pmatrix} \dfrac{df}{f}-\dfrac{dw}{w} \\ \dfrac{dw}{w}-d\omega \\ d\omega \end{pmatrix}.$$

Therefore

$$d^2\Phi \ge \delta\Big(\frac{1}{2}+\frac{2}{\alpha(\alpha-1)}+\frac{1}{\alpha}\Big)^{-1} x^{-\alpha}\frac{f^2}{w}\Big[\big(\frac{df}{f}-\frac{dw}{w}\big)+\big(\frac{dw}{w}-d\omega\big)+d\omega\Big]^2 = \delta' x^{-\alpha}\frac{(df)^2}{w}. \quad \square$$

References

[1] R. Hunt, B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for conjugate function and Hilbert transform, *Trans. Amer. Math. Soc.,* **176** (1973), pp. 227–251.

[2] S. Treil and A. Volberg, *Wavelets and the angle between past and future*, Univ. of Göttingen preprint, Geometrie und Analysis, heft 7 (1995), 1–33; to appear in Journal of Functional Analysis.

[3] D. Zheng, *The distribution function inequality and products of Toeplitz operators and Hankel operators*, to appear in Journal of Functional Analysis.