Large solutions for Yamabe and similar problems on domains in Riemannian manifolds

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We are going to present new results on the question of existence and uniqueness of positive solutions of the equation

$$\Delta u + hu - k\psi(u) = -f \quad \text{in } \Omega, \quad u(x) \to \infty \text{ as } \delta(x) = \operatorname{dist}(x, \partial\Omega) \to 0$$

on domains Ω with nonempty nonsmooth boundary in Riemannian manifolds M of dimension $n = \dim M \geq 2$. Here Δ is the Laplace-Beltrami operator given by the Riemannian metric on M. The function ψ is assumed to be well defined on all nonnegative numbers, vanishing at zero, increasing, convex and growing sufficiently fast as $u \to \infty$. As a typical example we can take $\psi(u) = u^\alpha$ for some $\alpha > 0$.

This equation with $\psi(u) = u^{(n+2)/(n-2)}$ arises in the problem of conformal change of metric in dimensions 3 and more and is known as the Yamabe problem. Let g, g' be two conformally related Riemannian metric. The conformal relationship will be written as $g' = u^{4/(n-2)}g$. Denote by R', R their scalar curvature functions. These are related by the equation

$$\Delta u - \frac{n-2}{4(n-1)}Ru + \frac{n-2}{4(n-1)}R'u^{(n+2)/(n-2)} = 0.$$

Here Δ is the Laplace-Beltrami operator in the metric g. Clearly this equation is a special case of the general problem, provided and $R' < 0$. In this light the Yamabe equation with boundary data $u(x) \to \infty$ as $x \to \partial\Omega$ can be seen as a problem of finding *complete* metric g' in Ω with given nonnegative scalar curvature R' such that g' is conformally related to the background metric g in M. The most typical example we want to consider is when R' is constant and negative. The meaning of the words *complete metric* is that all geodesics of g' in Ω never intersect the boundary $\partial\Omega$ of Ω, i.e., (Ω, g') is geodesically complete.

We present answers to both existence and uniqueness on a very general class of domains such as domain with fractal boundaries etc.