1/25/07 Comparison of infinite sets, cardinality, and countable sets

Set $X \cup Y$ $X \cap Y$

$x \in X$ $A \subset X$ ← A is a subset of X (A is a collection of elements)

↑ x is an element of X

EX $X = \{1,2,3,4,5\}$

$1 \in X$, $3 \in X$, $7 \notin X$

$\{1,3\} \subset X$, $\{1\} \subset X$

Function $\{X, Y, f\}$

↖↑ sets

X is the domain of the function (a set of arguments)

Y is the target space (range)

f is a rule that $\forall x \in X$ assigns a unique $y = f(x) \in Y$

EX
1. $X = \mathbb{R}$, $Y = \mathbb{R}$, f s.t. $f(x) = x^2$ $\forall x \in \mathbb{R}$
2. $X = [0,\infty)$, $Y = \mathbb{R}$, f s.t. $f(x) = x^2$ $\forall x \in [0,\infty)$
3. $X = [0,\infty)$, $Y = [0,\infty)$, f s.t. $f(x) = x^2$ $\forall x \in [0,\infty)$

} distinct functions

4. X = students in this class

   $Y = \mathbb{N}$

   f(student) = SISD #

Notation: $f : X \to Y$

<u>Def</u> $f : X \to Y$ is called injective (1 to 1, 1-1)

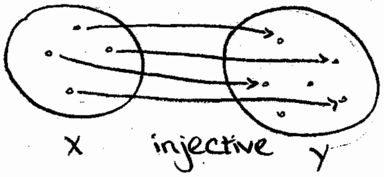
if $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Examples 2,3,4 above are injective

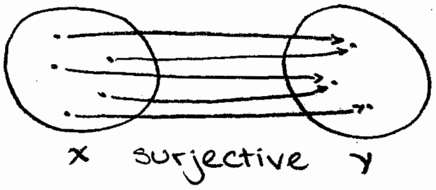
<u>Def</u> f is called surjective (if it takes every value $y \in Y$) ← informal

if $\forall y \in Y$ $\exists x \in X$ s.t. $f(x) = y$

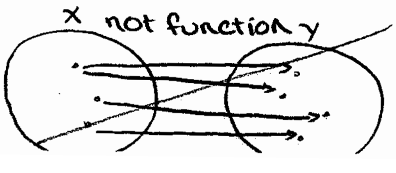
Example 3 above is surjective



injective



surjective



not function

Def $f: X \to Y$ is a bijection (1-1 correspondence) if f is injective and surjective

If f is a bijection, we can define an inverse function f^{-1}
If $y = f(x)$ then $x = f^{-1}(y)$

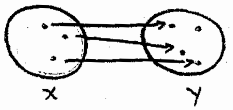
Cardinality = number of elements in a set

Def For sets X, Y, card X = card Y if $\exists$ bijection $f: X \to Y$
(f^{-1} is then a bijection $Y \to X$)

EX $\{1, 2, 3, \dots, n\} \overset{f \text{ bijection}}{\longleftrightarrow} X$ then X has exactly n elements (card $X = n$)

Def If card X = card $\mathbb{N}$, then X is called countable (countably infinite)

Def card $X \leq$ card Y if $\exists$ injection $f: X \to Y$

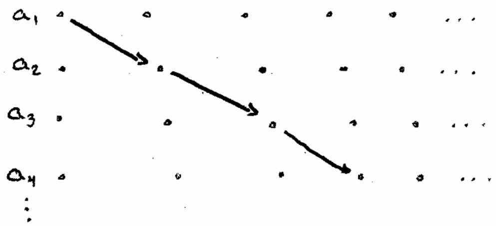


card $X \leq$ card Y

Thm (Bernstein-Schröder)
If card $X \leq$ card Y and card $Y \leq$ card X
then card X = card Y

Thm (Cantor)
The set of all sequences of 0 and 1 is uncountable

Pf List all sequences.



Take the diagonal and invert all elements (0 becomes 1, 1 becomes 0). The result is a sequence not on the list. Then the set of all such sequences is uncountable.

More formal PF

$$\begin{array}{llllll} 1^{st} \text{ sequence} & a_{11} & a_{12} & a_{13} \ldots & a_{1n} \ldots & \\ 2^{nd} & a_{21} & a_{22} & a_{23} \ldots & & \\ & a_{31} & a_{32} & a_{33} \ldots & & \nwarrow \text{DS (dyadic sequence)} \\ & \vdots & & & & \end{array}$$

$$\text{Define } b_n = \begin{cases} 0 \text{ if } a_{nn}=1 \\ 1 \text{ if } a_{nn}=0 \end{cases}$$

$$(b_n = a_{nn}+1 \pmod 2)$$

$\{b_n\}$ is not on the list

$\exists$ a natural bijection between DS and the collection of all subsets of $\mathbb{N}$

2^X is the collection all subsets of X

$a_1 a_2 \ldots a_n \ldots \leftrightarrow A = \{n : a_n \neq 0\}$

Card DS $=$ card $2^{\mathbb{N}} \neq$ card $\mathbb{N}$

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$2^X = \{$collection of all subsets of X

$\{1, 2, 3\}$

2^X $\{1\}, \{2\}, \{3\},$
$\{1,2\}, \{1,3\}, \{2,3\}$
$\{1,2,3\}, \phi$ (there are 2^3 elements)

<u>Ex.</u> Difference between $\underset{\substack{\uparrow \\ \text{set containing} \\ \text{the number one}}}{\{1\}}$ and $\underset{\substack{\uparrow \\ \text{the number} \\ \text{one}}}{1}$

$X \times Y$ - Cartesian product
$= \{(x,y) : x \in X, y \in Y\}$

<u>Ex.</u> $\mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ (xy coordinate plane)

<u>Thm.</u> Any infinite subset of $\mathbb{N}$ is countable.

<u>Pf.</u> $\underset{\substack{\uparrow \\ \text{infinite}}}{E} \subset \mathbb{N}$

n_1 first # in E
n_2 first # in $E \setminus \{n_1\}$
n_3 first # in $E \setminus \{n_1, n_2\}$

1. E is infinite, so we never stop (so we use all $\mathbb{N}$)
2. If $m \in E$, we get m in $\leq m$ steps

So we constructed a bijection from $\mathbb{N} \to E$

Remark: A sequence is a particular case of a function

Corollary: Infinite subset of a countable set is countable

$$\begin{tikzcd}
X \arrow[r, mapsto, "f : \text{bijection}"] & \mathbb{N} \\
\cup & \\
\underset{\text{infinite}}{A} \arrow[r, mapsto] & \underset{\text{infinite}}{f(A)} = \{ n \in \mathbb{N} : \exists x \in A, n = f(x) \}
\end{tikzcd}$$

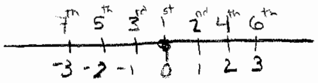
Thm. $\mathbb{N} \times \mathbb{N}$ is countable

Pf. All pairs:

$$\begin{array}{llll}
{}^{1}(1,1) & {}^{3}(1,2) & {}^{6}(1,3) & \ldots \\
{}^{2}(2,1) & {}^{5}(2,2) & {}^{9}(2,3) & \ldots \\
{}^{4}(3,1) & {}^{8}(3,2) & (3,3) & \ldots \\
{}^{7}(4,1) & & &
\end{array}$$

Thm. $\mathbb{Z}$, $\mathbb{Q}$ are countable

Pf. $\mathbb{Z}$:



$\mathbb{Q}$: For positive rationals, use the same method as $\mathbb{N} \times \mathbb{N}$, throwing out repeats as they come

or

1. $\mathbb{Z} \times \mathbb{Z}$ is countable (so we can work with $\mathbb{N} \times \mathbb{N}$)
2. We can identify $\mathbb{Q}$ with an infinite subset of $\mathbb{Z} \times \mathbb{Z}$

$r = \frac{p}{q}$ p, q have no common factor
sign p = sign r

3. $\mathbb{Q}$ is countable as an infinite subset of a countable set

Corollary of $\mathbb{N} \times \mathbb{N}$ is countable

Countable or finite union of countable sets is countable

Prop. Any infinite set contains a countable set

January 31 2007

$A \Rightarrow B$

$A \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \dots A_n \Rightarrow B$ Must justify each implication.

Known properties of countable sets:

1. Any subset of a countable set is either countable or finite.
2. Union of finitely many countable sets is countable.
3. Cartesian Product of 2 countable sets is countable.

3'. Countable union of countable sets is countable

3, 3' are equivalent: If C_k countable sets $k = 1, 2, 3, \dots$

then $\bigcup_{k=1}^{\infty} C_k$ countable. $C_1 = \{a_{11} \ a_{12} \ a_{13} \dots \}$ This produced the

$C_2 = \{a_{21} \ a_{22} \ a_{23} \dots \}$

$\vdots$

same table used to prove statement 3.

Prop Any infinite set contains a countable one.

Proof: Let X be the infinite set. $\exists x_1 \in X$, $\exists x_2 \in X, x_2 \neq x_1$

$\exists x_3 \in X \ x_3 \neq x_2, x_3 \neq x_1, \dots \ \exists x_n \in X \ x_n \neq x_k \ \forall k < n$

So we have $\{x_1, x_2 \dots x_n \dots\} \subset X$

Corollary: If X is infinite, C countable then $\text{card}(X \cup C) = \text{card}\, X$.

assume X is countable $\{x_1, x_2 \dots x_n \dots\}$. $\{x_1 \ x_2 \dots x_n \dots\}\{c_1, c_2 \dots c_n \dots\}$

There is a bijection from X to $(X \cup C)$.

Construction of real numbers:

$\mathbb{N}$ are taken to be given. $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$

real number as an infinite decimal fraction.

example: $0.999... = 1.000$, so representation is not unique.

forbid infinite sequences of zeroes only. Problems arise when adding, multiplying real numbers.

May say that real numbers are infinite binary fractions:

$0.a_1a_2... = \sum_{k=1}^{\infty} a_k 2^{-k}$ $a_n \in \{0,1\}$.

<u>Dedekind</u>: x - real $L = \{r \in \mathbb{Q} : r < x\}$

$L^c = \mathbb{Q} \setminus L = \{r \in \mathbb{Q} : r \geq x\}$

<u>Def</u> $L \subset \mathbb{Q}$ is called a Dedekind cut if

1. $\forall x \in L, \forall y \in \mathbb{Q}$ $y < x \Rightarrow y \in L$
2. L has no maximal element.
3. $L \neq \emptyset$
   $L \neq \mathbb{Q}$

Upper bound and maximal element.

A - collection of numbers. Number U is called an upper bound of A if $\forall a \in A$ $a \leq U$. U does not have to be unique or belong to set A.

Maximal element of A is $m \in A$ s.t. $\forall a \in A$ $a \leq m$.

A set may have an upper bound but not max element.

Max element of A = An upper bound belonging to A

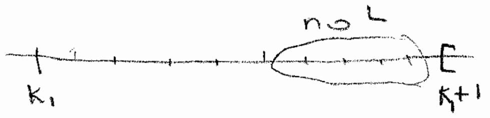
February 2, 200

Properties of Dedekind Cuts (cont'd)

3. $L \neq \phi$, $L \neq \mathbb{Q}$ (non-trivial)

4. There is one-to-one correspondence between Dedekind cuts and infinite decimal (binary) fractions.

1. If we have inf. dec. (bin.) fraction x, we can find all $q \in \mathbb{Q}$ $q < x$
2. If we have a cut L, find minimal $k \in \mathbb{Z}$ s.t. $[k+1, \infty) \cap L = \phi$



$k_1 \geq 0$

$j_1 = \min\{j \in \{0,1,\dots,9\} : [k_1 + \frac{j+1}{10}, \infty) \cap L = \phi\}$

$k.j_1$ (one decimal place obtained)

$j_2 = \min\{j \in \{0,1,\dots,9\} : [k_1 + \frac{j_1}{10} + \frac{j+1}{100}, \infty) \cap L\} = \phi\}$

↳ can also get binary decomposition ($j \in \{0,1\}$)

Arithmetic & Comparison of cuts

ⓧ+: $L_1 + L_2 = \{q_1 + q_2 : q_1 \in L_1, q_2 \in L_2\}$

ⓧ<: $L_1 < L_2 \Leftrightarrow L_1 \subsetneqq L_2$ (L_1 is a proper subset of L_2)

L corresponds to rationals iff $L^c = \mathbb{Q} \setminus L$ has minimal element (L corresponds to this minimal element)

Defns.

$\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$

$\mathbb{Q}_- = \{q \in \mathbb{Q} : q < 0\}$

$\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$ $(0 \Leftrightarrow \mathbb{Q}_-)$

Notations

$L^+ = L \cap \mathbb{Q}_+$

$\bar{L} = L^c \setminus \{$minimal element of L^c
if min. exists

ⓧ $L_1, L_2 > 0$ $L_1^+ \neq \phi$, $L_2^+ \neq \phi$

$(L_1) \cdot (L_2) = \{q_1 \cdot q_2 : q_1 \in L_1^+, q_2 \in L_2^+\} \cup \mathbb{Q}_- \cup \{0\}$

$-1 \cdot (L) = \{q : q \in \bar{L}\}$

Completeness property of $\mathbb{R}$

$E \subset \mathbb{R}$

Def E is bounded above if it has an upper bound, i.e.
$\exists u \in \mathbb{R} \ \forall x \in E \ x \leq u$

Def The smallest upper bound of E is called the least upper bound (supremum) of E $\sup E$

Thm Any bounded above $E \subset \mathbb{R}$ has the least upper bound.

Proof

E: collection of Dedekind cuts

$\bigcup L = \sup E$
$L \in E$

(can be proved using inf. dec. fractions, by localizations)
as in the beginnings of lect.

$u = \sup E$ if u is upper bound of E ①$(\forall x \in E, x \leq u)$
and $\forall u_1 < u \ \exists x \in E$ s.t. $x > u_1$ ②$(\forall \varepsilon > 0 \ \exists x \in E$ s.t. $x > u - \varepsilon)$

E is bounded below if $\exists l \in \mathbb{R}$ s.t. $\forall x \in E \ x \geq \underset{\substack{\uparrow \\ \text{lower bound}}}{l}$

The greatest lower bound is called infimum of E

$\inf E = -\sup(-E)$

where $-E = \{-x : x \in E\}$

Ex $E = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k} \dots\}$

$\sup E = 1$
$\inf E = 0$

$\mathbb{R}$ is a complete ordered field.

complete ordered field is unique.
(see handout about complete ordered fields)

$\boxed{\text{PAGE 1}}$ MA 101 notes by Orion Haunstrup

Lecture #
2/5
(my birthday

Limit of a Sequence

$\mathbb{N} \to \mathbb{R}$: $a_1, a_2, a_3, \dots, a_n$ or $\{a_n\}_1^\infty$ or $\{a_m\}_{n=m}^\infty$

Def: $a = \lim_{n\to\infty} a_n$ if $\forall\, \varepsilon > 0,\ \exists\, N = N(\varepsilon)$ s.t. $\forall\, n > N,\ |a_n - a| < \varepsilon$

$\uparrow$ (N is dependant on ε)

You can think of this definition as being like a "game with the devil"

textbook has a slightly different definition, but they are actually equivalent if you look closely

$\Rightarrow$ can use j for n, m for N

if $N(\varepsilon_0)$ works, (if N is great enough to satisfy ε_0))

then $N(\varepsilon)$ works $\forall\, \varepsilon > \varepsilon_0$

$\forall\, n > N(\varepsilon_0),\ |a_n - a| < \varepsilon_0 < \varepsilon$

In the definition of Lim, WLOG we can assume ε is rational or aka $\varepsilon = 1/M$ or $\varepsilon = 2^{-M}$, where $M \in \mathbb{N}$

$\uparrow$ "without loss of Generality"

Devil gives $\varepsilon > 0$. Pick $\varepsilon_0 = 1/M$, $M \in \mathbb{N}$ s.t. $\varepsilon_0 < \varepsilon$. Now find $N(\varepsilon_0)$

Assume ε is small. $\varepsilon < 1$, $\varepsilon < 10^{-6}$, $\varepsilon < \varepsilon_0$

Not all sequences have Limits!!

ex: $a_n = (-1)^n$ $\Rightarrow$ sequence has no Limit

☆ HW exercise $\Rightarrow$ prove that sequence $a_n = (-1)^n$ has no Limit (using definition of a Limit) ☆

what does it mean that a is not a Limit of $\{a_n\}_{n=1}^\infty$?

negation of Definition of Limit: $\exists\, \varepsilon > 0$ s.t. $\forall\, N,\ \exists\, n > N$ s.t. $|a_n - a| \geq \varepsilon$

we must show that $\forall\, a \in \mathbb{R}$, a is not a Limit $\lim_{n\to\infty} (-1)^n$

Consider the cases $a = 1$ and $a \neq 1$

$\downarrow$

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Ex: $\lim_{n\to\infty} \frac{1}{n^2} = 0$ let's prove it using definition of limit

given $\varepsilon > 0$, What is $N = N(\varepsilon)$? $\Rightarrow$ $N = \frac{1}{\sqrt{\varepsilon}}$ (or anything larger $\Rightarrow$ do not need to find the smallest one)

Ex: $\lim_{n\to\infty} \frac{1}{n^2+64n+15} = 0$

given $\varepsilon > 0$ $\Rightarrow$ you can still use $N = \frac{1}{\sqrt{\varepsilon}}$, as $\frac{1}{n^2+64n+15}$ will be smaller than $\frac{1}{n^2}$

$$\left\{ \frac{1}{n^2} < 1 \div \left(\frac{1}{\sqrt{\varepsilon}}\right)^2 = \varepsilon \quad \text{if } n > \frac{1}{\sqrt{\varepsilon}} \right\}$$

$\lim_{n\to\infty} \frac{1}{n^2-64n-15}$

suppose we proved that $15 < \frac{1}{3}n^2$ and $64n < \frac{1}{3}n^2$

$\hookrightarrow \forall n > \sqrt{45}$ $\hookrightarrow \forall n > 64 \cdot 3 = 192$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n^2-64n-15} < \lim_{n\to\infty} \frac{1}{1/3\, n^2}$, $\forall n > \max(\sqrt{45}, 192)$

$\frac{3}{n^2} < \varepsilon \Rightarrow n^2 > \frac{3}{\varepsilon} \Rightarrow n > \frac{\sqrt{3}}{\sqrt{\varepsilon}} \Rightarrow N = \max(\sqrt{45}, 192, \sqrt{\frac{3}{\varepsilon}}$

Thm: if a limit exists, it is unique

aka if $a = \lim a_n$ & $b = \lim a_n$, then $a = b$

Proof:

Let $a \neq b$. Put $\varepsilon = |a-b|/3$

(1) $a = \lim a_n \Rightarrow \exists N_1 \ni \forall n > N_1,\ |a - a_n| < \varepsilon$

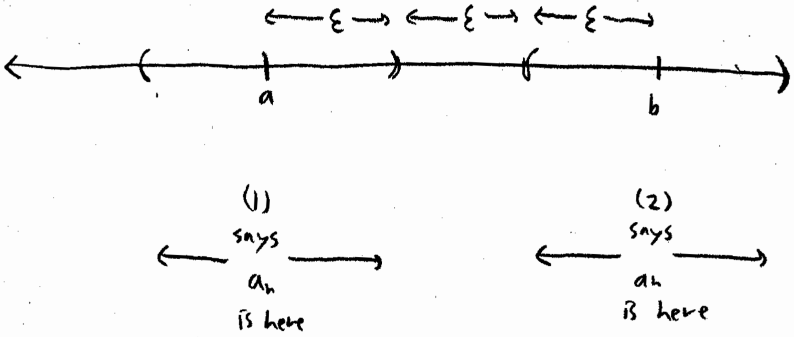
(2) $b = \lim b_n \Rightarrow \exists N_2 \ni \forall n > N_2,\ |b - a_n| < \varepsilon$

$N = \max(N_1, N_2) \Rightarrow$ both (1) and (2) are true

(continued...)

PAGE 3

2 ↓



pick $n > N$

$$3\varepsilon = |a-b| = |a-a_n+a_n-b| \leq |a-a_n| + |a_n-b| < \varepsilon + \varepsilon < 2\varepsilon$$

$$3\varepsilon < 2\varepsilon \wedge \varepsilon > 0 \Rightarrow 3 < 2 \Rightarrow \text{contradiction!}$$

$|x+y| \leq |x| + |y|$ is the triangle inequality

HW: go over online handout

2/7/07

Def. $a_n \nearrow$ if $\forall n$, $a_n \le a_{n+1}$

$a_n \searrow$ if $\forall n$, $a_n \ge a_{n+1}$

a_n strictly $\nearrow$ if $\forall n$, $a_n < a_{n+1}$

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e.g. $\forall n, a_n = 1$, $a_n \nearrow$ & $a_n \searrow$

★Thm: Let $\{a_n\}_{n=1}^{\infty}$ be increasing & bounded above

[$\exists A$ s.t. $\forall n$ $a_n \le A$]

Then $\lim a_n = \sup\{a_n : n \in \mathbb{N}\}$

Similarly:

Thm: $a_n \searrow$ & bounded below

Then $\lim a_n = \inf\{a_n : n \in \mathbb{N}\}$

$\{a_n\}_{n=1}^{\infty} \leftarrow$ sequence

$\{a_n : n \in \mathbb{N}\} \leftarrow$ set of values

$\sup\{a_n\}$, we mean $\sup = \{a_n : n \in \mathbb{N}\} \Longleftarrow$ supremum of the set

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Pf of ★: let $a = \sup a_n = \sup\{a_n : n \in \mathbb{N}\}$

- Upper bounded, so $\forall n < \mathbb{N}$ $a_n \le a$

- $\forall \varepsilon > 0$

  By definition $a - \varepsilon \neq \sup$, or upper bounded

$\Longrightarrow$

so $\exists N$, s.t. $a_n > a-\varepsilon$

then $\forall n > N$

$a-\varepsilon \le a_N \le a_n \le a < a+\varepsilon$

$\Rightarrow \quad a-\varepsilon < a_n < a+\varepsilon$

$|a_n - a| < \varepsilon$ ■

observation : $a_n \searrow$ and bounded below,

then $-a_n \nearrow$ and bounded above.

X, s.t. $-X = \{-x : x \in X\}$

then $\inf X = -\sup(-X)$

Def. $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence

if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n, k \in \mathbb{N}$, $n, k > N$

$\Rightarrow \quad |a_n - a_k| < \varepsilon$

Thm: If $\underline{\lim a_n}$ exists,

then $\{a_n\}$ is a Cauchy sequence.

$\underline{\underline{\text{Pf.}}}$ $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$ $|a_n - a| < \varepsilon/2$

Then $\forall n, k > N$

$$|a_n - a_k| = |a_n - a + a - a_k| \le |a_n - a| + |a - a_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ ■

Thm: If $\{a_n\}_{n=1}^{\infty}$ is a ~~Ch~~ Cauchy sequence of real,

then $\exists\ a \in \mathbb{R}$, $a = \lim_{n\to\infty} a_n$

Thm: (Pinching principle , Squeeze Theorem , 2 cops theorem)

let $a_n \le b_n \le c_n$, $\forall n \geq N_0$

If $\lim a_n = a = \lim c_n$

then $\lim b_n = a$

Def: $\{a_n\}$ (odd)

$$\overline{a}_n = \sup\{a_k : k \geq n\}$$

$$\underline{a}_n = \inf\{a_k : k \geq n\}$$

~~a~~ $\underline{a}_n \le a_n \le \overline{a}_n$

$\Rightarrow$ $\overline{a}_n \geq \overline{a}_{n+1}$ $\quad \overline{a}_n \searrow$

$\underline{a}_n \le \underline{a}_{n+1}$ $\quad \underline{a}_n \nearrow$

$$\exists \lim \overline{a}_n = \limsup_{n\to\infty} a_n = \overline{\lim_{n\to\infty}}\ a_n$$

$$\lim \underline{a}_n = \liminf_{n\to\infty} a_n = \underline{\lim_{n\to\infty}}\ a_n$$

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liminf & limsup always exist for bounded sequence

Obs: $\overline{\lim} = \underline{\lim} = a$

then $\lim a_n = 0$

Thm: TFSAE

1. $\lim_{n\to\infty} a_n$ exists
2. $\{a_n\}$ is a cauchy sequence
3. $\limsup a_n = \liminf a_n$ (finite)

We have already proved that 1 implies 2
and that 3 implies 1 (which follows from 2 cop theorem)

Goal for today: Prove that 2 implies 3

$$\begin{tikzcd} ① \arrow[r, Rightarrow] & ② \arrow[r, Rightarrow] & ③ \arrow[ll, bend left] \end{tikzcd}$$

$\boxed{2 \Rightarrow 3}$

Step 1: **Observation:** $\{a_n\}$ - cauchy then $\{a_n\}$ is bounded
$(\exists R \text{ s.t. } \forall n \; |a_n| \leq R)$

$\varepsilon = 1 \quad \exists N \quad \forall n, k > N \quad |a_n - a_k| < 1$

Pick $\overset{(\exists)}{\text{some}}$ $n > N$ $\quad R = \max\{|a_1|, |a_2|, \dots, |a_{n-1}|, |a_n| + 1\}$

Claim: $|a_k| \leq R \quad \forall k \in N$

why? $k < n \quad a_k \leq \max\{|a_1|, \dots, |a_{n-1}|\}$
$k \geq n \quad |a_n - a_k| < 1$
so $|a_k| < |a_n| + 1$

favorite trick: $|a_k| = |a_k - a_n + a_n|$
$\leq |a_n| + |a_k - a_n|$
$< |a_n| + 1$

Step 2: Take $n \in N$
Consider the following $\sup\{a_j - a_k : j, k \geq n\}$

$$\sup\{a_j - a_k : j, k \geq n\} \overset{(*)}{=} \sup\{a_j : j \geq n\} - \inf\{a_k : k \geq n\}$$
$$= \bar{a}_n - \underline{a}_n$$

(*) - assume for a moment that this equality is true

$\{a_n\}$ Cauchy $\Rightarrow \forall \varepsilon>0 \quad \exists N = N(\varepsilon) \quad \forall j,k>N \quad |a_j-a_k|<\varepsilon$

s.t. $\forall n>N, \quad 0 \leq \bar{a}_n - \underline{a}_n = \sup\{\underbrace{a_j-a_k}_{<\varepsilon} : j,k \geq n\} \leq \varepsilon < 2$

$\bar{a}_n - \underline{a}_n < \varepsilon$

Conclusion: $\underbrace{\lim \; (\bar{a}_n - \underline{a}_n)}_{} = 0$

$\underbrace{\lim \bar{a}_n}_{(\limsup)} - \underbrace{\lim \underline{a}_n}_{(\liminf)}$

Still have to prove (*)

$X, Y \subset \mathbb{R} \qquad X+Y = \{x+y : x \in X, y \in Y\}$

$\underbrace{k}_{\mathbb{R}} X = \{kx : x \in X\}$

$X - Y = X + (-1)Y = \{x-y : x \in X, y \in Y\}$

__Thm:__ $\sup(X+Y) = \underbrace{\sup X}_{a} + \underbrace{\sup Y}_{b}$

__Pf:__ – $\forall x \in X \quad \forall y \in Y \quad x \leq a, y \leq b$ so $x+y \leq \underbrace{a+b}_{\text{u.b. for } X+Y}$

$\sup(X+Y) \leq a+b$

– Take $(\forall)$ $\varepsilon>0 \quad \exists x \in X : x > a - \frac{\varepsilon}{2}$ – (not u.b. by def of sup)

$\exists y \in Y : y > b - \frac{\varepsilon}{2}$ – (not u.b)

$x+y > a+b-\varepsilon$ so $a+b-\varepsilon$ not u.b. for $X+Y$

Conclusion: $a+b$ u.b. for $X+Y \quad \forall \varepsilon>0 \quad a+b-\varepsilon$ not

so $a+b = \sup X+Y$ Q.E.D

To prove (*)

<u>Obs:</u> $\inf X = -\sup(-X)$

$$\sup\{a_j : j \geq n\} - \inf\{a_k : k \geq n\}$$
$$= \sup\{a_j : j \geq n\} + \sup\{-a_k : k \geq n\}$$
$$= \sup\{a_j - a_k : j, k \geq n\}$$
$$\uparrow$$

because $\sup(X+Y) = \sup X + \sup Y$

<u>Remark:</u> $\sup_n (a_n + b_n) \leq \underset{\parallel}{\sup a_n} + \sup b_n$ (looks like a contradiction but is <u>not</u>!)

$$\sup\underbrace{\{a_n + b_n : n \in \mathbb{N}\}}_{\cap \;\; \neq \;\leftarrow \text{generally}} = \sup\underbrace{\{a_n : n \in \mathbb{N}\}}_{X} + \sup\underbrace{\{b_n : n \in \mathbb{N}\}}_{Y}$$

$$X + Y = \{a_n + b_k : n, k \in \mathbb{N}\}$$

For Problem #5 –

counterexample that: $\limsup a_n + b_n \neq \limsup a_n + \limsup b_n$

2/12/07

Infinite limits & Extended Real line

Def: $\lim_{n\to\infty} a_n = \infty$
if $\forall\ R \in \mathbb{R}\ \exists N\ \ \forall n > N$
$a_n > R$

$\lim a_n = -\infty$
same as above except $a_n < R$

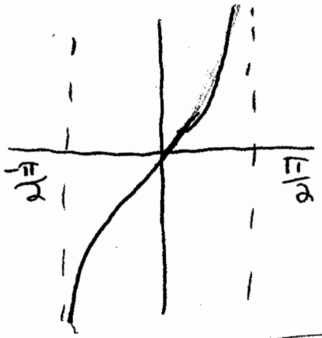
$A \subset \mathbb{R}$ not bdd. above then $\sup A = \infty$
$A \subset \mathbb{R}$ not bdd. below then $\inf A = -\infty$

Any $A \subset \mathbb{R}$ has sup & inf if we admit $\pm\infty$
$\liminf (a_n)$ and $\limsup (a_n)$ always exist if admit $\pm\infty$.

$\lim a_n = \infty$ iff $\limsup a_n = \liminf a_n = \infty$

Attention: if $\lim a_n = \infty$ then a_n not Cauchy

$\varphi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$
$\varphi(x) = \tan x$



Thus, bijection between open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$
$[-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ close the interval by adding $\pm\infty$ to $\mathbb{R}$

Using Cauchy seq., we can prove existence of limit without computing it.

Complete metric spaces.

Completion

Metric space: some set equipped with distance
X $\quad$ d - distance
$d : X \times X \to \mathbb{R}$

1) $d(x,y) \geq 0\ \forall x, y \in X$
2) $d(x,y) = 0 \Leftrightarrow x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in X$

Ex 1) $\mathbb{R}$, $\mathbb{Q}$:

$$d(x,y) = |x-y|$$

2) $\mathbb{R}^2$ $x=(x_1,x_2)$
$y=(y_1,y_2)$

$$d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

3) $\mathbb{R}^3$
$\mathbb{R}^n$

$$\sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}$$

4) distance between words (how many letters need to be changed).

cop	cup	1
cop	con	1
cop	policeman	8

Instead of $|a_n-a|<\varepsilon$, use $d(a_n,a)<\varepsilon$ in metric space.

In metric space, define Cauchy seq. the same way except use $d(a_n,a_k)<\varepsilon$ instead of $|a_n-a_k|<\varepsilon$

Def: A metric space X is complete if all Cauchy sequences have limit.

$\mathbb{R}$ is a complete metric space

Let X be a metric space

$\exists\ \overline{X}\supset X$, s.t. 1) $\overline{X}$ is complete m.s.
2) X is dense in $\overline{X}$

2) means
$\forall x\in\overline{X}$ $\exists$ seq. $\{x_n\}$
$x_n\in X$ $\forall n$ and $\lim x_n = x$

Ex: $\mathbb{R}$ is a completion of $\mathbb{Q}$

Remark Completion is "unique":
If $\overline{X}$, $\overline{Y}$ are 2 completions of X, then $\exists$ bijection f.
$f:\overline{X}\to\overline{Y}$; 1) $d(x,y)=d(f(x),f(y))$ $\forall x,y\in\overline{X}$
2) $f(x)=x$ $\forall x\in X$

Standard Construction of Completion:

elements of $\bar{X}$ are Cauchy Sequences in X

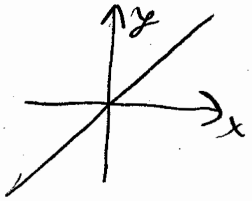
$\{\frac{1}{n}\} \leftrightarrow 0$ Representation by C.S. are not unique.

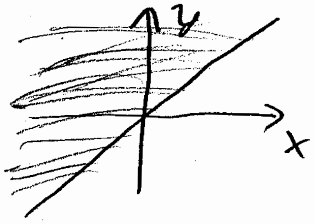
$\{\frac{1}{n^2}\} \leftrightarrow 0$

Equivalence relations & quotient spaces

Ex. of relations: $=$, $\leq$, $>$, etc. on $\mathbb{R}$

Relation on X is a subset of $X \times X$





X, relation $\sim$

$x \sim y$

$\sim$ is called equivalence if: $\forall x, y, z \in X$

1) $x \sim y \Rightarrow y \sim x$ (symmetric)
2) $x \sim x$ (reflexive)
3) $x \sim y, y \sim z \Rightarrow x \sim z$ (transitive)

Ex: $=$

$\parallel$ (parallel lines)

congruent

similar Δ

$\frac{a}{b} \sim \frac{c}{d} \Rightarrow ad = bc$

X, $\sim$;

$\forall x \in X$ define $\{y \in X : y \sim x\}$

Two sets: A B

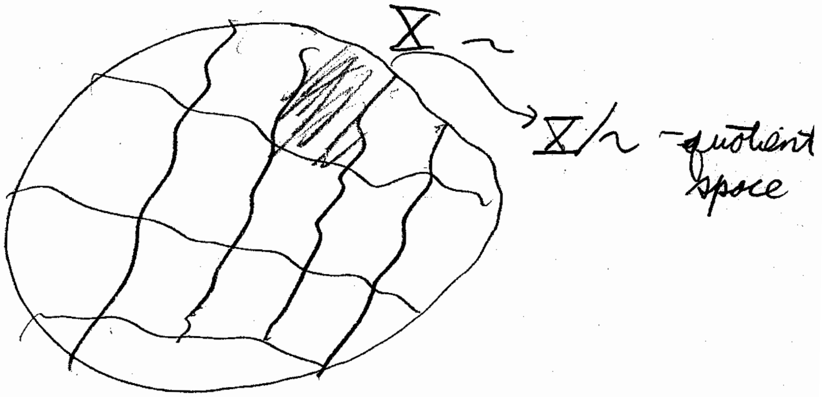
x y

two possibilities:

1) $A \cap B = \phi$

or

2) $A = B$



Note taker: Bruno Feb 14 07

$X \sim$ ↑ equivalence relation splits set

coset aka equivalence class

$X/\sim$ quotient space

X - metric

$\mathcal{X}$ - collection of all Cauchy seq $\{x_n\}_1^\infty$ $x_n \in X$

$\{x_n\} \sim \{y_n\}$ if $\lim d(x_n, y_n) = 0$

$\bar{X} = \mathcal{X}/\sim$

X is a subset of $\bar{X}$

$x \in X$
$\downarrow$
$x, x, x, \ldots \in \mathcal{X}$

metric on $\bar{X}$
$d(\{x_n\}, \{y_n\})$
$= \limsup d(x_n, y_n)$

Arithmetic of limits

Thm $\lim(a_n + b_n) = \lim a_n + \lim b_n$ if $\lim a_n$ $\lim b_n$ exist.

$$\boxed{\text{wrong: } \lim a_n, \lim b_n \nexists \Rightarrow \lim(a_n + b_n) \nexists}$$

Pf. $\forall \varepsilon > 0$

$a = \lim a_n \Rightarrow \exists N_1 \; \forall n > N_1 \; |a - a_n| < \varepsilon/2$

$b = \lim b_n \Rightarrow \exists N_2 \; \forall n > N_2 \; |b - b_n| < \frac{\varepsilon}{2}$

$\forall n > N \quad |a_n + b_n - (a+b)| \leq |a_n - a| + |b_n - b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thm If $\lim a_n = a$ then $\lim c \cdot a_n = ca = c \lim a_n$

Prove as exercise.

Thm. If $\{a_n\}$ is bounded $(\exists R \text{ s.t } \forall n \; |a_n| \leq R)$ and $\lim b_n = 0$ then $\lim a_n \cdot b_n = 0$

Proof as H.W exercise

Thm. If $\lim a_n = a$ $\lim b_n = b$ then $\lim a_n b_n = ab$

L. If $\lim a_n$ exists, then $\{a_n\}$ is bounded.

$\lim a_n$ exists $\Rightarrow \{a_n\}$ cauchy $\Rightarrow \{a_n\}$ bounded

Pf $a_n = a + \alpha_n$ $\qquad \alpha_n = a - a_n$

$b_n = b + \beta_n$

$$\lim \alpha_n = \lim_{n \to \infty} (a_n - a) = 0$$

$\lim \beta_n = 0$

$$(a+\alpha_n)(b+\beta_n) = ab + \underset{\substack{\downarrow \\ 0}}{\alpha_n} b + \underset{\substack{\downarrow \\ \text{bounded}}}{\alpha_n} \underset{\substack{\downarrow \\ 0}}{\beta_n} + a \underset{\substack{\downarrow \\ 0 \\ n \to \infty}}{\beta_n} = ab$$

$a_n = \underline{O}(\frac{1}{n})$

means $\exists \, c$ s.t $|a_n| \leq c \frac{1}{n} \; \forall n$

$\{a_n\} = \underline{O}(1)$

$\{b_n\} = \overset{=}{o}(1)$

$$(a+\alpha_n)\underbrace{(b+\beta_n)}_{b_n} = \underset{\substack{\downarrow \\ ab}}{a \cdot b_n} + \underset{\substack{\downarrow \\ 0}}{\alpha_n b_n}$$

1. Subsequences, Limit points of a sequence

$a_1, a_2, \dots, a_n, \dots$

$n_k \cdot k = 1, 2, \dots, n, \dots$ e.g. 1, 5, 48

$n_k < n_{k+1}$ a_1, a_5, a_{48}

a_n a_{n_k} b_1, b_2, b_3 ← subsequence

Thm/ If $a = \lim a_n$, then $\forall$ subsequence a_{n_k}, $\lim_{k\to\infty} a_{n_k} = a$

Consider all convergent subsequences, limit point = limit of a convergent subseq.
(having limit)

Thm/ If a_n bounded above then $\exists\, a_{n_k}$ s.t. $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$

just a relaxation of earlier thm

Pf/ HW

Thm/ Any bounded sequence a_n has a convergent subseq. a_{n_k} (*) important

Prop/ If $\{a_n\}$ bounded then $\limsup a_n$, $\liminf a_n$ are maximal & minimal limit points.

2. Topology of real line, open & closed sets (3.2)

(a,b) ← open $[a,b]$ ← closed

$a < x < b$ $a \leq x \leq b$

this definition works for any metric space

Def/ $A \subset \mathbb{R}$ is called open if $\forall x \in A\ \exists \varepsilon > 0$ s.t. $B(x,\varepsilon) = \underbrace{\{y : \overbrace{|y-x|}^{d(x,y)} < \varepsilon\}}_{(x-\varepsilon,\, x+\varepsilon)} \subset A$

Textbook definition: $\forall x \in A\ \exists a, b$ s.t. $x \in (a,b) \subset A$
(non-symmetric)

Thm/ 1. Union of any collection of open sets is open.

2. If $A_1, A_2, \ldots A_n$ are open, then $\bigcap_{k=1}^{n} A_k$ is open

Pf/ 1. $\mathcal{A}$ - collection of open sets

$$\bigcup_{A \in \mathcal{A}} A = \{x : \exists A \in \mathcal{A} \text{ s.t. } x \in A\}$$

NB: this works for infinite sets

Take $x \in \bigcup_{A \in \mathcal{A}} A$

Take (arbitrary, $\exists$) $A \in \mathcal{A}$, $x \in A$ - open

$\exists\, \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset A \subset \bigcup_{A \in \mathcal{A}} A$

Pf/ 2. Let $x \in \bigcap_{k=1}^{n} A_k$

$\forall k \in \{1, 2, \ldots n\}$ $x \in A_k$ so

$\exists\, \varepsilon_k$ s.t. $B(x, \varepsilon_k) \subset A_k$

Define $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots \varepsilon_k\}$

then $\forall k \in \{1, \ldots, n\}$

$B(x, \varepsilon) \subset B(x, \varepsilon_k) \subset A_k$

$\therefore B(x, \varepsilon) \subset \bigcap_{k=1}^{n} A_k$

HW Show that $\exists$ open A_k s.t. $\bigcap_{k=1}^{\infty} A_k$ not open

(construct counterexample for statement 2 under infinite sets)

(a, b) - open

$\mathbb{R}$ - open

$\emptyset$ - open (by definition)

$(0, 1]$ - not open. why? $\exists x = 1$ $\forall \varepsilon > 0$, $B(1, \varepsilon) \not\subset (0, 1]$

HW p. 98 #1

p. 107 #6 (open sets)

-21

open sets (cont.)

Def Neighborhood of x
= An open set $A \ni x$ } 'open neighborhood'

nota bene: Sometimes a neigh. of x is any $A \supset \underset{\overset{\uparrow}{\text{open}}}{U} \ni x$ (in other literature)

$a = \lim a_n$ if $\forall$ neigh $U \ni a$ $\exists N$ s.t. $\forall n > N$ $a_n \in U$ (*)

Classical Def.:

'neigh' becomes $B(a,\varepsilon) = (a-\varepsilon, a+\varepsilon)$, which is more specific ('special'), and therefore stronger. If N can be found for $\forall$neigh, then it can be found for the specific set $B(a,\varepsilon)$

Equivalence of Defs: (from classical)

Let $\underset{\text{neigh}}{U} \ni a$ $\exists \varepsilon > 0$ st $B(a,\varepsilon) \subset U$ (Def of open set)

Find $N = N(\varepsilon)$ (classical) $\forall n > N$ $|a - a_n| < \varepsilon \Leftrightarrow a_n \in B(a,\varepsilon)$

$\Rightarrow a_n \in U$

* (*) does not require a metric

Def $A \subset \mathbb{R}$, $x \in \mathbb{R}$, x is an accumulation point (limit point) of A if any neighborhood U of x contains inf many points of A

Remark Equivalent Def: x is accumulation point of A iff $\forall$neigh of x, U, $A \cap (U \setminus \{x\}) \neq \emptyset$

EX: $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

set of accumulation points: $\{0\}$

$A = (0,1)$, accumulation points $= [0,1]$

$A = \mathbb{Q}$, acc points $= \mathbb{R}$

Def A is <u>closed</u> if A^c is open

<u>Thm</u> A is closed iff:

A contains all of its accumulation points

<u>Thm</u> A_α is closed $\rightarrow \bigcap_\alpha A_\alpha$ is closed

<u>Thm</u> $A_1, \dots, A_n$ - closed

$\rightarrow \bigcup_{k=1}^{n} A_k$ is closed

to prove:

De Morgan's laws:

$$\left(\bigcup_\alpha A_\alpha\right)^c = \bigcap_\alpha A_\alpha^c$$

$$\left(\bigcap_\alpha A_\alpha\right)^c = \bigcup_\alpha A_\alpha^c$$

Pf:

open because A_α is closed

Show $\cap A_\alpha$ is closed: $(\cap A_\alpha)^c = \bigcup_\alpha \widehat{A_\alpha^c}$ is closed because it is union of open

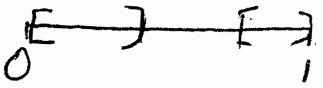
so $(\bigcap_\alpha A_\alpha)^c$ is open and $\bigcap_\alpha A_\alpha$ is closed

! "A is not open, so A is closed" is <u>WRONG</u>! $(-1,1]$ is neither open nor closed

Examples of closed sets;

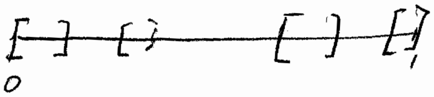
1. $[a,b]$
2. $\mathbb{R}$. $\Big\}$ both open and closed
3. $\emptyset$

$\frac{1}{3}$ Cantor set



$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

repeat for C_2



C_2

$$\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$$

February 23,200
Joyce Kwok

Thm A closed, $a_n \in A \ \forall n$
and $a = \lim_{n\to\infty} a_n$ ($\lim_{n\to\infty} a_n$ exists and is equal to a)
Then $a \in A$

Proof

A closed $\Leftrightarrow A^c$ open

Let $a \notin A$, i.e. $a \in A^c$

A^c open, so $\exists \varepsilon > 0$ s.t. $B(a,\varepsilon) \subset A^c$

By defn of limit, $\exists N = N(\varepsilon)$
s.t. $\forall n > N \ |a_n - a| < \varepsilon$ $\quad (\underset{\underset{A}{\cap}}{a_n} \in B(a,\varepsilon))$

$$\Rightarrow \underset{\underset{a_n \ \forall n > N}{\cup}}{B(a,\varepsilon)} \cap A \neq \phi$$

$\Downarrow$ (contradiction)
$\Uparrow$

$B(a,\varepsilon) \subset A^c \Leftrightarrow B(a,\varepsilon) \cap A = \phi.$

Continuous functions

$\neq$ functions that you can draw without lifting your hand

1) Dirichlet function

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

2) $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \cos nx$

} functions that you cannot simply draw.

Informal Defn small error in argument
$\Rightarrow$ small error in result

Formal Defn $f : \underset{\underset{\mathbb{R}}{\cap}}{D} \to \mathbb{R}$

Let $x_0 \in D$
We say that f is continuous at x_0
if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
$\forall x \in D \ |x - x_0| < \delta \Rightarrow |f(x_0) - f(x)| < \varepsilon$

Last line can be restated in terms of sets:
$f(B(x_0, \delta) \cap D) \subset B(f(x_0), \varepsilon)$
$\hookrightarrow f(A) = \{f(x) = x \in A\}$ takes all possible values of A

Remark Defn is interesting (non-trivial) only if x_0 is an accumulation point of D.

If x_0 is not an accumulation point of D then any $f : D \to \mathbb{R}$ is continuous at x_0.

(since $f(B(x_0,\delta)\cap D)$ is just the point x_0, $f(x_0) \subset B(f(x_0),\varepsilon)$)

$f : \underset{\substack{\uparrow \\ \text{isolated points}}}{\mathbb{Z}} \to \mathbb{R}$ is continuous at any $x \in D = \mathbb{Z}$

Example 1 $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

$\mathbb{R} \to \mathbb{R}$

not continuous at any $x_0 \in \mathbb{R}$

(HW)

Ex 2 $f : \mathbb{Q} \to \mathbb{R}$

$f(x) = 1 \;\; \forall x \in \mathbb{Q}$

Continuous $\forall x_0 \in \mathbb{Q}$

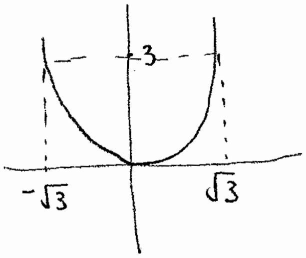
Defn $f : D \to \mathbb{R}$ is continuous if f is continuous at all $x_0 \in D$.

Notation Inverse image (pre-image)

$$f^{-1}(\underset{\substack{\uparrow \\ \text{set}}}{A}) = \{x \in D : f(x) \in A\}$$

Ex $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$

$f^{-1}([0,3]) = [-\sqrt{3}, \sqrt{3}]$



$f^{-1}((0,3)) = (-\sqrt{3}, 0) \cup (0, \sqrt{3})$

$f^{-1}((-\infty,3)) = (-\sqrt{3}, \sqrt{3})$

Thm Let $f: \mathbb{R} \to \mathbb{R}$

Then f is continuous iff

$\forall$ open $A \subset \mathbb{R}$

$f^{-1}(A)$ open

Proof

1. Let f be continuous.

Take open A and let $\underline{\underline{\overset{(\forall)}{x_0}}} \in f^{-1}(A)$

$f(x_0) \in A$ open $\underset{\text{defn of open}}{\Rightarrow} \exists \varepsilon > 0$ s.t. $B(f(x_0), \varepsilon) \subset A$.

f is continuous at x_0

So $\underline{\underline{\exists \delta > 0}}$ s.t. $f(B(x_0, \delta)) \subset B(f(x_0, \varepsilon)) \subset A$

$\Rightarrow \underline{\underline{B(x_0, \delta) \subset f^{-1}(A)}}$ (defn of open set)

2. other direction will be proved in a later lecture

Corollary $f: \mathbb{R} \to \mathbb{R}$

f cont. iff $\forall$ closed A $f^{-1}(A)$ closed

HW

1. Dirichlet function

2. $f: \mathbb{R} \to \mathbb{R}$
construct examples s.t.
f cont. but
   1. $f(A)$ not open
   $\exists A$ open s.t.
   2. $\exists$ closed B $f(B)$ not closed

3. P.139 #13

Note by Jin H.

2/28/07

Advice for Homework: · Write clearly, Use space

1st Midterm: Two weeks from today, Wed., March 14

From the textbook:

$$f : \mathcal{D} \longrightarrow \mathbb{R}$$

$\downarrow$

open

f cont. iff $\forall$ open $A \subset \mathbb{R}$
$f^{-1}(A)$ open

If $\mathcal{D}$ is open,
then A is open in $\mathcal{D}$ iff A is open in $\mathbb{R}$

A open in $\mathcal{D}$ means $\exists$ open U in $\mathbb{R}$,
s.t. $A = \mathcal{D} \cap U$ is open

- Connected

1) Is $\mathbb{Q}$ connected?

→ No.

pf: $\sqrt{2}$

$A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$
$B = (\sqrt{2}, -\infty) \cap \mathbb{Q}$ $\Big\}$ open in $\mathbb{Q}$

2) $\mathbb{R} \setminus \mathbb{Q}$?

→ No

3) Any interval in $\mathbb{R}$ is connected.

Revisiting Def.

- Connected $\mathcal{D}$: $\mathcal{D}, \phi$ the only open & close sets.
- Not Connected : $\exists$ a non-trivial open & closed in $\mathcal{D}$ set A

$$A, \quad B = \mathcal{D} \setminus A$$
open & close

$$\underset{\text{open}}{\mathcal{D} = A \cup B} \qquad A \cap B = \phi$$

Limit of function. Connection with sequence

Def || $f : \mathcal{D} \to \mathbb{R}$ $\quad x_0$ — accumulation point of $\mathcal{D}$

We say $a = \lim_{x \to a} f(x)$

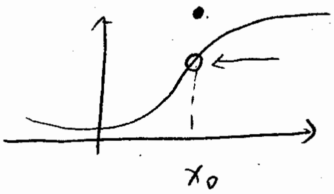
if $\exists\, \varepsilon > 0$, $\exists\, \delta > 0$, s.t. $\forall\, x \in \mathcal{D}$,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < \varepsilon$$

We did not assume $x_0 \in \mathcal{D}$.

so it's possible to have :

EX.



$$\forall\, B(a, \varepsilon),\ \exists\, B(x_0\ \delta),\ \text{s.t. } f[B(x_0, \delta) \cap \mathcal{D} \setminus x_0] \subset B(a_0, \varepsilon)$$

☆ THM : f is continuous at x_0 iff $f(x_0) = \lim_{x \to x_0} f(x)$

(assuming $x_0 \in \mathcal{D}$ ~~In~~ is an accumulation point of $\mathcal{D}$)

☆ THM: $f: \mathcal{D} \to \mathbb{R}$, x_0 - acc point of $\mathcal{D}$

$a = \lim_{x \to x_0} f(x)$ iff $\forall$ sequence $\{x\}_{n=1}^{\infty}$, $x_n \neq x_0$ $\forall n \in \mathbb{N}$, $x_n \in \mathcal{D}$

$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = a$

Remark: such seq $\{x_n\}$ exist because x_0 is acc. point.

Pf: $\Rightarrow$

Let $a = \lim_{x \to x_0} f(x)$

Take $\{x\}$ as stated in thm.

Take $\varepsilon > 0$ $(\forall)$

$\exists \delta > 0$, s.t. $\forall x \in \mathcal{D}$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < \varepsilon$

$\Uparrow$ def of limit of function.

Now use def. of sequence:

$\exists N = N(\varepsilon)$, s.t. $\forall n > N$, $|x_0 - x_n| < \delta$, (δ: replace ε in the definition)

assume $|x_n - x_a| > 0$, $x_n \neq x_a$

$\therefore$ $\forall n > N$

$|f(x_a) - a| < \varepsilon$

This is the definition of limit $\longrightarrow \lim_{n\to\infty} f(x_n) = a$

$A \Rightarrow B$

Negation would be:

$\neg B \Rightarrow \neg A$ (contrapositive to $A \Rightarrow B$)

$\boxed{\Leftarrow}$ Assume $a \neq \lim_{x \to x_0} f(x)$ (Negation of the result)

Pf will be provided next lecture.

E.X.: ① $\lim_{x \to 0} \sin \frac{1}{x}$

Pick $x_n \to 0$ $(\lim x_n = 0)$

$x_n \neq 0$ $\forall n$

s.t. $f(x_n)$ not Cauchy.

Pf: /HW

② $\lim_{x \to 0} x \sin \frac{1}{x}$

find limit & prove your result. /HW

P13 #2, 12 ← From last assignment

3-2-07

- Proof of connection between limit of sequence and limit of function (Part 2)

want to construct $x_n \to x_0$ $x_n \neq x_0 \forall n$
$\lim_{n\to\infty} x_n = x_0$, s.t. $\lim_{n\to\infty} f(x_n) \neq a$

(contrapositive: $a \Rightarrow b \leftrightarrow \neg b \Rightarrow \neg a$)

assume $a \neq \lim_{x\to x_0} f(x)$

$\exists\ \varepsilon > 0,\ \forall \delta > 0,\ \exists\ x \neq x_0$ s.t. $|x - x_0| < \delta$ & $|f(x) - a| \geq \varepsilon$

$\delta_n : \delta_n \searrow 0$. (eg $\delta_n = \frac{1}{n}$ or $\frac{1}{2^n}$)

$\forall \delta_n, \exists\ x_n$ s.t. $|x_0 - x_n| < \delta_n$ & $|f(x_n) - a| \geq \varepsilon$ $\quad x_n \neq x_0 \forall n$

$\therefore \lim_{n\to\infty} x_n = x_0$ (if given $\alpha > 0$, I find N: $\delta_n < \alpha$ then $\forall n > N$ $|x_0 - x_n| < \delta_n \leq \delta_N < \alpha$)

$|f(x_n) - a| \geq \varepsilon$ $\forall n$ so $\lim f(x_n) \neq a$

- Composition of Functions

f, $\mathcal{D}(f)$ domain $\quad g$, $\mathcal{D}(g)$ domain

f is function
f(x) is value of f at point x

$\forall x \in \mathcal{D}(g)$ $g(x) \in \mathcal{D}(f)$

$f(g(x)) = f \circ g$

for example: $x \mapsto \sin^2 x$ $\quad f \circ g$ $\quad g(x) = \sin x$ $\quad f(x) = x^2$

- Thm: f, g $f \circ g$ defined. x_0 - acc pt. of $\mathcal{D}(g)$. $\lim_{x\to x_0} g(x) = y_0$.
let $y_0 \in \mathcal{D}(f)$ & f is cont. at x_0.
Then $\lim_{x\to x_0} f\circ g(x) = \lim_{x\to x_0} f(g(x)) = f(y_0) = f(\lim_{x\to x_0} g(x))$

e.g.: $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \ln x} = e^{\lim_{x\to 0^+} x \ln x}$
(then use L'Hospital)

- Cor: f, g $f \circ g$ defined. f, g cont.
Then $f \circ g$ cont.

(can also specify at a specific point)

- Continuous functions:
$1, x \to$ polynomials
rational (when defined)
$|x|$, $\sin x$, e^x

- Thm: f, g cont. $\quad f \vee g(x) = \max\{f(x), g(x)\}$
Then $f \vee g$ is continuous

Proof:
$f \vee g = \frac{|f - g| + f + g}{2}$ $\quad f \geq g \Rightarrow \frac{f - g + f + g}{2} = f$ $\quad g \geq f \Rightarrow \frac{g - f + f + g}{2} = g$
addition, subtraction, abs. value & scalar mult. all preserve continuity
$\therefore f \vee g$ continuous.

- Thm: f, g cont. $f \wedge g = \min(f, g)$
Then $f \wedge g$ continuous.

3/5/07 Mon.

Thm (Intermediate value Thm)

Let f be continuous on $[a,b]$, and $f(a)\cdot f(b) < 0$

Then $\exists\ c \in (a,b)$ s.t. $f(c)=0$

proof) $a_1 = a$, $b_1 = b$, given a_n & b_n (s.t. $f(a_n)\cdot f(b_n) < 0$), let $C_n = \frac{a_n+b_n}{2}$

Construct $C_1 = \frac{a_1+b_1}{2}$

(i) $f(c_1) = 0$: end of proof

(ii) $f(a_1)\, f(c_1) > 0$: $f(c_1)\, f(b_1) < 0$. Put $a_2 = c_1$ $b_2 = b_1$

(iii) $f(b_1)\cdot f(c_1) > 0$: $f(a_1)\, f(c_1) < 0$ Put $a_2 = a_1$ $b_2 = c_1$

repeat the same procedure...

If lucky, we will get $f(c_n) = 0$

If not lucky, $a_n \nearrow$, $b_n \searrow$ on $[a,b]$ (both bounded & monotone)

$f(a_n)$ have the same sign

$f(b_n)$ have the same sign

$|b_n - a_n| = \frac{b-a}{2^{n-1}}$, $\lim a_n$ and $\lim b_n$ $\exists$.

$$\lim(b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^{n-1}} = 0$$

$$\therefore \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \underline{\underline{k}}$$

WLOG, assume $f(a) > 0$, $f(b) < 0$

Then, $f(a_n) > 0\ \forall n$, $f(b_n) < 0\ \forall n$.

Because $\lim a_n = c$ and f is continuous,

$$f(c) = \lim_{n\to\infty} a_n \geqslant 0$$

Similarly

$$f(c) = \lim_{n\to\infty} b_n \leqslant 0$$

$\therefore$ $f(c) = 0$, Q.E.D.

Because $\lim_{n\to\infty} a_n = C$ and f is continuous,

$$f(c) = \lim_{n\to\infty} a_n \geqslant 0$$

Similarly,

$$f(c) = \lim_{n\to\infty} b_n \leqslant 0$$

$\therefore\ f(c) = 0$. Q.E.D.

Thm Any interval is connected.

I: If $a, b \in I$, then $\forall x,\ a < x < b \Rightarrow x \in I$

(Idea of the proof)

Given an interval I, $\underbrace{A, B}_{\downarrow}$ (none empty sets) s.t. $A \cup B = I$, $A \cap B = \phi$

two open & closed in I

take $a_1 \in A$, $b_1 \in B$

Construct an increasing sequence $c_1 = \frac{a_1 + b_1}{2}$

s.t. $\lim a_n = \lim b_n = \lim c_n$

$c_n = \frac{a_n + b_n}{2}$ If $c_n \in A$ then $a_{n+1} = c_n$, $b_{n+1} = b_n$.

If $c_n \in B$ then $a_{n+1} = a_n$, $b_{n+1} = c_n$

Assume $c \in A$. $\exists \varepsilon$ s.t. $B(c,\varepsilon) \cap I \subset A$

$B(c,\varepsilon) \cap I \cap B = \phi$ $|b_n - c| \geqslant \varepsilon$

this contradicts with $\lim_{n\to\infty} b_n = c$. $\therefore$ C does not belong to A

Similarly, C does not belong to B.

Thm f continuous, A connected. then $f(A)$ connected.

Cor Let p be a polynomial of odd degree. Then $\exists x_0 \in \mathbb{R}$ s.t. $p(x_0) = 0$

proof) $p(x) = a_{2n+1} x^{2n+1} + a_{2n} x^{2n} + \cdots + a_0$, $a_{2n+1} \neq 0$, assume $a_{2n+1} > 0$ WLOG.

$\lim_{x \to +\infty} p(x) = +\infty$ $\exists b$ s.t. $p(b) > 0$

$\lim_{x \to -\infty} p(x) = -\infty$ $\exists a < b$ s.t. $p(a) < 0$

Intermediate Value Theorem. $p(b) \cdot p(a) < 0$. $\therefore \exists x_0 \in \mathbb{R}$ s.t. $p(x_0) = 0$. QED.

HW p138 #3, 4, 7, 8, 17

↑
ignore question
about open intervals

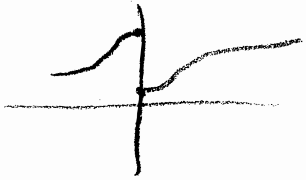
Any cont. funct. has IVP

— IVP - $\forall a<b$ $f(x) : x \in (a,b)$ takes all values between $f(a)$ and $f(b)$

c between $f(a), f(b)$

$F(x) = f(x) - c$

$F(a), F(b) < 0$



$\lim_{x \to x_0^+} f(x)$ $\qquad f : D \to \mathbb{R}$

Restrict f onto $D \cap (x_0, \infty)$

call it f_1

$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f_1$

- similar for $\lim_{x \to x_0^-}$

Compact Sets

Def. Set $K \subset \mathbb{R}$ called sequentially compact if $\forall$ seq $\{x_n\}$ $x_n \in K$, $\exists \{x_{n_k}\}$ s.t. $x_{n_k} \to x_0 \in K$

Thm. $K \subset \mathbb{R}$ compact iff. bounded and closed

— Not true for general metric spaces

Pf. K bdd, closed

take $\{x_n\}$, $x_n \in K$, $\forall n$ $|x_n| \le R_0$ so

$\exists x_{n_k} \to x_0 \in \mathbb{R}$

K is closed → contains all limits, $x_0 \in K$

K compact - if K unbounded then $\forall n$ $\exists x = x_n$ s.t. $|x_n| > n$

$\forall$ subseq x_{n_k} $\lim |x_{n_k}| = \infty$ - no finite limit

So K not unbounded if compact

If K not closed, $\exists \{x_n\}_1^\infty$ $\lim x_n = x_0 \notin K$

then $\forall$ subseq $\{x_{n_k}\}_1^\infty$, $\lim x_{n_k} = \lim x_n = x_0 \notin K$

Thm. $f : D \to \mathbb{R}$ $K \subset D$

f cont. on K, cont. at all points of K

$F(K) = \{ f(x) : x \in K \}$ compact

continuous image of compact domain compact

$F(K)$ compact for cont. f and compact K

Pf. Take $\{y_n\}$ $y_n \in F(K)$

$\exists x_n$ $x_n \in K$; $f(x_n) = y_n$

K - compact $\Rightarrow$ $\exists x_{n_k} \to x_0 \in K$

$\lim_{k \to \infty} y_{n_k} = \lim_{k \to \infty} f(x_{n_k}) = f(x_0)$

because f cont. at $x_0 \in K$

Thm: $\underset{\text{cont.}}{f} : \underset{\overset{\uparrow}{\text{compact}}}{K} \to \mathbb{R}$, then $F(K)$ is bounded

& $\exists x_{min}, x_{max}$ s.t. $f(x_{max}) = \sup \{f(x), x \in K\}$

$f(x_{min}) = \inf \{ f(x) : x \in K \}$

cont. function on compact set obtains max & min.

$f(x) = x$ on $(0,1)$ does not attain max.

$(0,1) \to \frac{1}{x}$ has no bound

A open, f cont. f(A) not open

B closed, f cont f(B) not closed – not true
counter example

About midterm

Definitions maybe
(which of these sets are open)-example
And other things

Homework

A, B closed

A+B not.

A, B compact, $\Rightarrow A+B$ compact

$$\begin{tikzcd} A \arrow[d] & B \arrow[d] \\ a_n & b_n \end{tikzcd}$$

$c_n = a_n + b_n$

$a_{n_k} \to a \in A$

$b_{n_k} \to b \in B$

$\Rightarrow c_n$ convergent.

$A = \mathbb{N}$

$A+B \supset \{\frac{1}{n}\ n \in \mathbb{N}\}$

$\not\supset \{0\}$ So it is not closed.

* * *

$f: K \to \mathbb{R}$

continuous complete

$\Rightarrow f$ attains sup and inf.

Pf

$f(K)$ compact $\Rightarrow f(K)$ bounded

$M := \sup f(K) = \sup\{f(x); x \in K$

$\varepsilon_n = \frac{1}{n}$

$M - \varepsilon_n$ is not Upper bound for $f(K)$
(there is a bigger one)

$\Rightarrow \exists\, x = x_n$ s.t. $f(x_n) > M - \frac{1}{n}$

$f(x_n) \leq M < M + \frac{1}{n}$

M is upper bound.

$\Rightarrow |f(x_n) - M| < \varepsilon_n = \frac{1}{n}$

So $\lim f(x_n) = M$

get ε, pick N s.t. $\varepsilon_N = \frac{1}{N} < \varepsilon$
$\forall n > N \quad |f(x_n) - M| < \frac{1}{n} < \frac{1}{N} < \varepsilon$

K compact $\Rightarrow \exists\, x_{n_k} \to x_0 \in K$

$$M = \lim_{n\to\infty} f(x_n) = \lim f(x_{n_k}) \underset{\text{because } f \text{ continuous,}}{=} f(x_0)$$

$f(x_{n_k})$ subseq. of $f(x_n)$
(limit of subseq. = lim of seq.)

Better proof of) would use — if $K \subset \mathbb{R}$ is compact $\Rightarrow \sup K \in K$, $\inf K \in K$ and then apply it to $f(K)$

gave us that proof because if you want to minimize something you have to prove that soln. exists, this is a usual way of doing it.

take something not exactly soln. but almost, take seq. of that, and find limit (if compact, there's a convergent subseq.)

Can't write formula maybe but can prove existence.

Uniform Continuity

Def: $f : D \to \mathbb{R}$, $D \subset \mathbb{R}$ for arbitrary metric space

f uniformly continuous

if $\forall \varepsilon > 0,\ \exists \delta > 0\ \forall x, y \in D\ \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

looks like defn. of continuity

[redacted] depends only on ε independent of x.

difference of where quantifier is

f cont. at x means

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in D\ \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$\delta(x, \varepsilon)$ depends on choice of x

f cont. means

$\forall x \in D\ \forall \varepsilon > 0,\ \exists \delta > 0\ \forall y \in D\ \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Continuous but not uniformly conts

ex $f:(0,1] \to \mathbb{R}$

$f(x)=\frac{1}{x}$

you can take ε = anything (it goes to ∞)

take $\varepsilon=1$

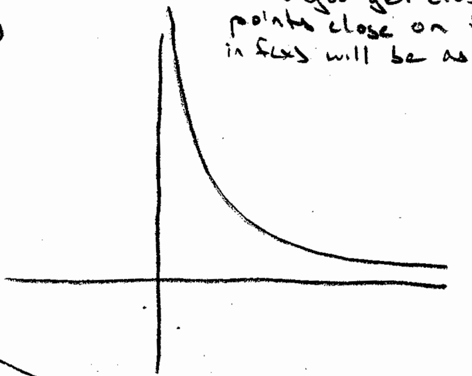
$(\forall)\ \delta>0$

take $x<\min(\delta,1)$

$y=\frac{x}{2}$

$\underset{f(y)}{\frac{1}{y}}-\underset{f(x)}{\frac{1}{x}}=\frac{2}{x}-\frac{1}{x}=\frac{1}{x}>1$

when you get close to 0, you can make the points close on the x-axis, but the difference in f(x) will be as big as you want.



(negation of unif. cont.)

Not uniformly continuous: $\exists\ \varepsilon>0\ \forall\ \delta>0\ \exists\ x,y\in D$ s.t. $|x-y|<\delta$, $|f(x)-f(y)|\geq\varepsilon$

$[0,\infty)\to\mathbb{R}$

$x\mapsto x^2$

$(0,\infty)\to\mathbb{R}$

$x\mapsto\sin(\frac{1}{x})$

not uniformly cont. (but cont.)

Important for integration proving cont. fxns are integratable

---

If $f:K\to\mathbb{R}$
cont. comp.

$\Rightarrow f$ is uniformly cont.

PF

Let f is continuous but not uniformly cont.

$\exists\ \varepsilon>0\ \forall\ \underset{\text{a for that }\delta}{\delta_n=\frac{1}{n}}\ \exists\ x_n,y_n\in K$ s.t. $\underbrace{|x_n-y_n|<\frac{1}{n}}$ and $|f(x_n)-f(y_n)|\geq\varepsilon$ contradict.

$\lim(x_n-y_n)=0$

$=\lim x_n-\lim y_n$

$\Rightarrow \lim x_n=\lim y_n=y_0 \Rightarrow$ by continuity $\lim f(x_n)=f(x_0)$, $\lim f(y_n)=f(x_0)$

but bad proof because limits might not exist. nothing tells you that $\lim x_n-\lim y_n$ exist

But it's ok if compactness

$x_{n_k} \to x_0 \in K$

can't say $y_{n_k} \to$

but take subseq of $y_{n_{k_j}} \to y_0$

$x_{n_{k_j}} \to x_0$

$\lim x_{n_{k_j}} - \lim y_{n_{k_j}} = \lim (x_{n_{k_j}} - y_{n_{k_j}}) = \lim (x_n - y_n) = 0$

$\Rightarrow x_0 = y_0$

so for proof you can say $\lim x_{n_{k_j}} - \lim y_{n_{k_j}}$

$\Rightarrow \lim x_{n_{k_j}} = \lim y_{n_{k_j}} = y_0$

Contradicts it,

$\lim f(x_{n_{k_j}}) = f(x_0)$

$\lim f(y_{n_{k_j}}) = f(x_0)$

3/12 ①

Heine-Borel Property

Set K $\quad$ $\mathcal{U}_\alpha$ is a cover

if $K \in \bigcup_\alpha \mathcal{U}_\alpha$

Thm

$K \subset \mathbb{R}$ is compact iff each open cover of K has finite subcover.

$\forall$ collection $\mathcal{U}_\alpha$ of open sets

If $K \subset \bigcup_\alpha \mathcal{U}_\alpha$ then $\exists\ \alpha_1, \alpha_2, \ldots, \alpha_n$

S.T. $K \subset \bigcup_{k=1}^{n} \mathcal{U}_{\alpha_k}$

Remark: TRue for Metric Spaces.

- Exam -

in class

7 problems.

Definitions. [limit, continuity, open/closed set etc...]

Logic. [Negations]

Examples. [open, closed, both; compact]

(no connected/unconnected)

IVT

Examples

$\{0\}$ - closed
compact

$\{\frac{1}{n} : n \in \mathbb{N}\}$ - not open
not closed
not compact

$\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ - closed, compact

$(0,1]$ - not open, not closed.

- 2 equivalent Defns of closed
- Limit of Seq.

  Prove(?) $\lim(x_n + y_n) = \lim(x_n) + \lim(y_n)$
- Limit of Functions. & continuity.

  Connections b/w lim of function & sequences.

  Connection b/w continuity & inverse images of open/closed sets
- Counter examples for Direct images of open & closed sets.

  $B \subset \mathbb{R}$ closed f cont. $f(B)$ not closed

  $\rightarrow$ $f(x) = \frac{1}{1+x^2}$ $B = [0, \infty)$ $f(B) = (0,1]$

3/16/07

Lecture 22

Derivatives

$f : D \to \mathbb{R}$

$x_0 \in D$

x_0 is an interior point, i.e. $\exists\ B_\varepsilon(x_0) \subset D$

Def f is differentiable at x_0 if $f(x_0+h) = f(x_0) + a \cdot h + r(h)$

where $\lim\limits_{h\to 0} \dfrac{|r(h)|}{|h|} = 0$

If $\exists\ a \in \mathbb{R}$ and $r(h)$ s.t. $\lim\limits_{h\to 0} \dfrac{|r(h)|}{|h|} = 0$
and $f(x_0+h) = f(x_0) + ah + r(h) \quad \forall\ |h| < \varepsilon$, then
f is differentiable

This "a" is called the derivative of x_0 $\qquad a = f'(x_0)$

Usual definition

$$\lim_{h\to 0} \frac{f(x_0+h)+f(x_0)}{h} = \lim_{h\to 0} \frac{a\cdot h + r(h)}{h} = \underbrace{\lim_{h\to 0} \frac{ah}{h}}_{a} + \underbrace{\lim_{h\to 0} \frac{r(h)}{h}}_{0} = a$$

Remark

r in the definition of derivative depends on x_0 $\quad (r_{x_0}(h))$

Take f, g defined in a neighborhood of x_0 or defined on D s.t. x_0 is an accumulation pt. of D

$f = O(g)$ if $\exists\ \varepsilon > 0,\ C < \infty$ s.t. $|f(x)| < C|g(x)| \quad \forall\ x$ s.t. $|x - x_0| < \varepsilon$

$\left|\dfrac{f(x)}{g(x)}\right| \le C \quad \forall x$ s.t. $|x - x_0| < \varepsilon$

$f = o(g)$ if $\lim\limits_{x\to x_0} \dfrac{|f(x)|}{|g(x)|} = 0 \qquad$ (so, if $f(x)=0$, then $g(x)=0$)

$f(x_0+h) = f(x_0) + f'(x_0)h + o(h)$

$f(x+h) = f(x) + f'(x)h + o(h)$
$g(x+h) = g(x) + g'(x)h + o(h)$

$$f(x+h)\cdot g(x+h) = f(x)\,g(x) + f(x)\,g'(x)h + g(x)\,f'(x)h$$

$$h \to 0 \qquad \boxed{\begin{array}{l} + f'(x)\,g'(x)\,h^2 + f'(x)\cdot h \cdot o(h) + o(h)\cdot g(x) + o(h)\,g(x)h \\ + o(h)\cdot o(h) + f(x)\cdot o(h) \end{array}} \Rightarrow \text{equal to } 0$$

$$f(x)\cdot o(h) = f(x)\cdot r(h) \text{ s.t. } \lim_{h\to 0} \frac{r(h)}{h} = 0$$

$$= \lim_{h\to 0} \frac{|f(x)\cdot r(h)|}{|h|} = 0$$

$\lim_{h\to 0} o(h) = 0$ because $\lim_{h\to 0} \frac{o(h)}{h}\cdot h = 0$ by product of limits

$o(h)\cdot o(h) = o(h^2)$

$o(h)\cdot O(h) = o(h^2)$

Same type of algebra can be done with $\left(\frac{f}{g}\right)'$

Derivatives and Continuity

f is differentiable implies that f is continuous

The converse is false

Ex. $|x|$

$\sqrt[3]{x}$

$\exists$ continuous function f on $\mathbb{R}$ s.t. $\forall x$ $f'(x) \nexists$

$$f(x) = \sum_{n=1}^{\infty} a_n \cos b_n x$$

$\sum |a_n|$ is finite

$b_n \nearrow \infty$ very "fast"

3/19/07

Remark about derivatives:

$f'(x_0)$: sufficient to assume x_0 is an accumulation pt. of $\mathcal{D}$

In 1 variable

May use $\mathcal{D} = [a,b]$, be asked to compute $f'(a)$, $f'(b)$, will be called 1-sided derivatives

---

Max, min & mean value theorem

Def: $f : \underset{\mathbb{R}}{\underset{\cap}{\mathcal{D}}} \to \mathbb{R}$

$x_0 \in \mathcal{D}$ is called a point of max (f has maximum at x_0)

if $\forall x \in \mathcal{D}$, $f(x_0) \geq f(x)$

↖ Global max

x_0 is a point of local max if $\exists \epsilon > 0$ s.t.

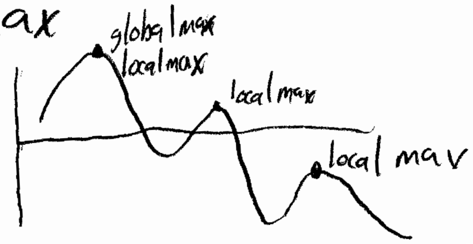
$f(x_0) \geq f(x)$ $\forall x \in \mathcal{D} \cap B(x_0,$

↑ $(x_0-\epsilon, x_0+\epsilon)$

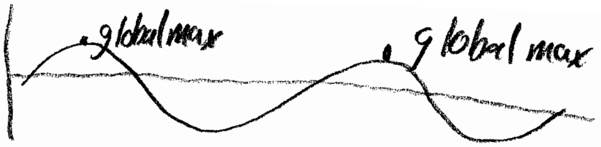
Note: For min def., just reverse signs.

x_0 is global max $\underset{\cancel{\Leftarrow}}{\Rightarrow}$ x_0 is local max

1.

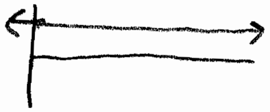


2. $f(x) = \sin x$



: infinite global max

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = 1$ $\forall$ x : Any point x_0 is a point of global max and min.



Note: Max, min often called extremum when which one is not important.

Thm: $f: \mathcal{D}_{\subset \mathbb{R}} \to \mathbb{R}$

Let x_0 be an interior point of $\mathcal{D}$:

$\exists \epsilon > 0$ $B(x_0, \epsilon) \in \mathcal{D}$

Let f have local extremum at x_0

Let $f'(x_0)$ exists.

Then $f'(x_0) = 0$

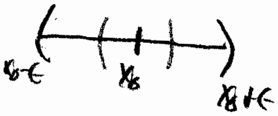
Pf: Pick $|h| < \epsilon$

$$f(x_0 + h) = f(x_0) + f'(x_0) \cdot h + r(h)$$

$$r(h) = o(h).$$

Consider a local max at x_0

$f(x_0) \geq f(x_0 + h)$ $\forall |h| < \delta \leq \epsilon$



Let $f'(x_0) \neq 0$

We know: $\lim_{h \to 0} \frac{|r(h)|}{|h|} = 0$

$\exists \delta_1 \leq \delta$ s.t. $\forall h$ $|h| < \delta_1 \Rightarrow \frac{|r(h)|}{|h|} < \frac{|f'(x_0)|}{2}$

$|r(h)| < \frac{1}{2} |f'(x_0) \cdot h|$

pick $|h| < \delta_1$ s.t. $f'(x_0) h > 0$

$f(x_0 + h) = f(x_0) + f'(x_0) \cdot h + r(h)$

$f(x_0 + h) \geq f(x_0) + |f'(x_0) \cdot h| - |r(h)|$

∴ continued:

$$f(x_0+h) \geq f(x_0) + |f'(x_0)\cdot h| - |r(h)|$$
$$\geq f(x_0) + \tfrac{1}{2} f'(x_0)h| > f(x_0) \quad \boxed{\Rightarrow\Leftarrow}$$

<u>Remark</u>: x_0 is interior point, was used to pick h: $f'(x_0)\cdot h > 0$

Theorem fails if x_0 is not int. pt.

Example: $f: [a,b] \to \mathbb{R}$

f has local max at b. and $f'(b)$ exists

then $f'(b)$ is positive, $(f'(b) \neq 0)$

From HW:

$$\lim_{n\to\infty} \underbrace{\lim_{k\to\infty} [\cos(\pi n! x)]^{2k}} = \begin{cases} 1 : & |\cos\ldots| = 1 \\ 0 : & |\cos\ldots| < 1 \end{cases}$$

$$|\cos(\pi n! x)| = 1 \iff xn! \in \mathbb{Z}$$

Note: when $x \in \mathbb{Q}$, $xn! \in \mathbb{Z}$

when $x \notin \mathbb{Q}$, $xn! \notin \mathbb{Z}$

Function becomes Dirichlet

Mean Value Theorem:

<u>Thm</u>: $f: [a,b] \to \mathbb{R}$, Let $f'(x)$ exist $\forall x \in (a,b)$

Assume f is continuous.

Then $\exists x_0 \in (a,b)$ s.t. $f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$

$$f(b) - f(a) = f'(x_0)\cdot(b-a)$$

Will prove next time.

Applications of MVT:

$$f'(x) \geq 0 \quad \forall x \in (a,b)$$

then f is increasing on (a,b)

$(f\uparrow)$

$x_1 < x_2 \in (a,b)$

Applying MVT, $f(x_2) - f(x_1) = \underbrace{f'(\underset{\uparrow (x_1,x_2)}{x_0})}_{\geq 0}(b-a) \geq 0$

$$\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right) = \lim_{x\to\infty}\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})}$$

but, if we have $\lim_{x\to\infty}\left((x+1)^p - x^p\right)$ $p \in (0,1)$

Use MVT, $f(x) = x^p$ $\quad f'(x) = px^{p-1}$

$$\underset{\uparrow b}{(x+1)^p} - \underset{\uparrow a}{x^p} = p \cdot \underset{\uparrow (x,x+1)}{x_0^{p-1}} = 1 \leq px^{p-1} \to 0$$

HW: 5.1, 5.2

pg 164 # 2, 6, 10, 11, 12

Michael Foley 3/21/07

Remark:

If f has $f'(x)$ $\forall x \in D$

$D \to R$

$x \mapsto f'(x)$

$x \mapsto f'(x)$ is cont. f is continuously differentiable on D

$f \in C^1(D)$

In MVT we do not assume that $f \in C^1((a,b))$, we only assume that $f'(x)$ $\exists$ for all $x \in (a,b)$

Thm: (Rolle's Theorem)

If $f \in C([a,b])$, $f'(x) \exists$ $\forall x \in (a,b)$ and $f(a) = f(b)$, then $\exists\, x_0 \in (a,b)$ s.t. $f'(x_0) = 0$

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Pf: $f \in C(\underset{\overset{\uparrow}{\text{compact}}}{[a,b]})$ so $\exists\, x_{max}, x_{min} \in [a,b]$

s.t. $f(x_{min}) \leq f(x) \leq f(x_{max})$

Case 1: $f(x_{max}) = f(x_{min}) = f(a) = f(b)$

then $f(x) = f(a)$ $\forall x \in [a,b]$

$f'(x) = 0$ $\forall x$

Case 2: either $f(x_{max}) > f(a) = f(b)$ or $f(x_{min}) < f(a) = f(b)$

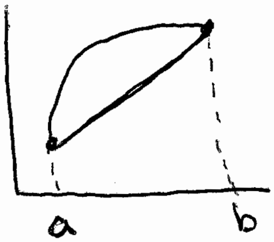
If $f(x_{max}) > f(a) = f(b)$

$\Rightarrow x_{max} \in (a,b)$

$\Rightarrow f'(x_{max}) = 0$

MVT Proof:

3/21/07



- Consider affine function L, $L(x) = mx + c$
  s.t. $L(a) = f(a)$, $L(b) = f(b)$

$L(x) = f(a) + m(x-a)$ $m = \frac{f(b)-f(a)}{b-a}$

$= f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$F = f - L$ $F \in C([a,b])$ $F'(x) \exists \; \forall x \in (a,b)$ $F(a) = F(b) = 0$

by Rolle's Theorem:

$\exists x_0 \in (a,b)$ s.t. $F'(x_0) = 0$

$$\overset{\shortparallel}{f'(x_0)} = m = \frac{f(b)-f(a)}{b-a}$$

Calculus of derivatives: s.5.3 → go over this by yourself

Taylor's Formula: next time

HW s.5.2, 5.3
pg. 164 #3, 13, pg. 176 #1, 2, 3

3/23/07

Taylor's formula

Def of derivative: $f(x_0+h)=f(x_0)+\underset{\nwarrow f'(x_0)}{a}\cdot h+o(h)$

$f(x_0+h)=f(x_0)+\underset{\uparrow f'(x_0)}{a_1}h+a_2h^2+\ldots+\underbrace{a_nh^n}_{\nwarrow \text{smallest term as } h\to 0}+o(h)^n \leftarrow$ Goal.

Assume f is a polynomial of degree n

Represent $f(x_0+h)=\sum_{k=0}^{n}a_kh^k$

This is possible because $a_k=a_k(x_0)$ is fixed, x_0 is fixed, and f is a polynomial so can be written as $\sum_{k=0}^{n}A_k(x_0+h)^k$

$\nwarrow$ expand and group powers of k

$f(x)=1+2x+7x^2-2x^3+3x^4-11x^5$

$x_0=1$

$x=(x_0+h)=(1+h)$

Let $h=0$. Then $f(x_0)=a_0$.

Take derivative in terms of h:

$\frac{d}{dh}: f'(x_0+h)=\sum_{k=1}^{n}ka_kh^{k-1}$

Plugging in $h=0$, $f'(x_0)=a_1$

Now take the second derivative:

$\frac{d^2}{dh^2}: f''(x_0+h)=\sum_{k=2}^{n}k(k-1)a_kh^{k-2}$

$f''(x_0)=2\cdot 1a_2 \quad \Rightarrow a_2=\frac{f''}{2!}$

$\frac{d^3}{dh^3}: f'''(x_0+h)=\sum_{k=3}^{n}k(k-1)(k-2)a_kh^{k-3}$

$f'''(x_0)=3\cdot 2\cdot 1a_3 \quad \Rightarrow a_3=\frac{f'''}{3!}$

$\boxed{a_k=\frac{f^{(k)}(x_0)}{k}}$

Then $f(x_0+h)\sim\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}h^k$

Note: $1!=1$ and $0!=1$

$(f^{(0)}=f)$

$T_n(h)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}h^k$

Recalling that $h=x-x_0$, $\overset{\swarrow \text{or } T_n(x)}{T_n(x-x_0)}=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

These are both forms of Taylor's polynomial of f of degree n at x_0

Remark $\frac{d^k}{dh^k}f(x_0+h)\Big|_{h=0}\overset{\swarrow\text{evaluated at}}{=}\frac{d^k}{dh^k}T_n(h)\Big|_{h=0} \quad \forall k=0,1,\ldots,n$

f is n times continuously differentiable

<u>Thm</u> (Taylor's theorem) Let $f \in C^n$ in a neighborhood of x_0. Then $f(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + o(h^n)$

If $n=1$, this is the definition of derivative.

PF (for $n=2$) of Taylor's theorem

Consider $F(h) = f(x_0+h) - \sum_{k=0}^{2} \frac{f^{(k)}(x_0)}{k!} h^k$

$F(0)=0$

$F'(0)=0$

$F''(0)=0$

Apply the MVT

$F(h) = F'(h_1)h$ for h_1 between $0, h$

(to be continued next class)

Applications of Taylor's theorem:

- Application to computing limits

$\lim_{h\to 0} \frac{\sin h}{h}$: $\sin h = 1 \cdot h + o(h)$ by Taylor's theorem

$$\lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{h+o(h)}{h} = 1$$

$\lim_{h\to 0} \frac{1-\cos h}{h^2}$: $(\cos x)' = -\sin x \quad \overset{x=0}{0}$

$(\cos x)'' = -\cos x \quad -1$

$$\lim_{h\to 0} \frac{1-\cos h}{h^2} = \lim_{h\to 0} \frac{1-(1+0\cdot h + \frac{-1}{2!}h^2 + o(h^2))}{h^2}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}h^2 + o(h^2)}{h^2} = \frac{1}{2}$$

$\lim_{h\to 0} \frac{1-\cos h^4}{h \cdot \tan^7 h}$ would be very tedious when solved by L'Hopital's rule but is simple with Taylor's formula

Homework: Pg. 176: # 3, 4, 5, 15
Pg. 192: # 2, 3, 8, 16

4/2

$$f(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + o(h^n)$$

$$F(h) = f(x_0+h) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k$$

$$F(0) = F'(0) = F''(0) = \cdots = F^{(n)}(0) = 0$$

MVT: $F(h) = F(h) - F(0) = h \cdot F'(h_1) \qquad h_1 \in (0,h)$

MVT: $F'(h_1) = F'(h_1) - F'(0) = h_1 \cdot F''(h_2) \qquad h_2 \in (0,h_1)$

$(*) = hh_1 F''(h_2) = hh_1h_2 F'''(h_3) \qquad h_3 \in (0,h_2)$

$= \ldots hh_1 \cdots h_{n-1} F^{(n)}(h_n)$

$h_k \in (0, h_{k-1})$

In particular, $|h_k| < |h|$

$$F(h) = \underbrace{hh_1 \cdots h_{n-1}}_{O(h^n)} \cdot \underbrace{(f^{(n)}(x_0+h_n) - f^{(n)}(x_0))}_{o(1)} \quad \text{that is,} \to 0 \text{ as } h \to 0$$

$= o(h^n)$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + o((x-x_0)^n), \quad h = x - x_0$$

<u>Thm</u> (Lagrange Remainder Theorem)

$f \in C^{(n+1)}((\alpha,\beta)) \qquad x_0 \in (\alpha,\beta)$

Then $(\forall x \in (\alpha,\beta)\ \exists \xi \in (x_0,x))$ $f(x-x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k$

$+ \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}, \quad \xi \in (x_0,x)$

Fix $x \neq x_0$

$$\varphi(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (t - x_0)^k - C(t - x_0)^{n+1}$$

$$\varphi(t)\big|_{t=x} = 0, \quad \varphi(x_0) = \varphi'(x_0) = \cdots = \varphi^{(n)}(x_0) = 0$$

Rolle's Theorem $\quad \varphi(x_0) = \varphi(x) = 0$

$\exists\, \xi_1 \in (x_0, x)$ s.t. $\varphi'(\xi_1) = 0$

$\exists\, \xi_2 \in (x_0, \xi_1)$ s.t. $\varphi''(\xi_2) = 0$

$\xi_k \in (x_0, \xi_{k-1})$ s.t. $\varphi^{(k)}(\xi_k) = 0 \qquad k = 1, 2, \ldots, n, n+1$

$$\varphi^{(n+1)}(t) = f^{(n+1)}(t) - 0 - (n+1)! \cdot C$$

$$\varphi^{(n+1)}(\xi_{n+1}) = 0$$

$$C = \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}$$

$$\varphi(x) = 0$$

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k - \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!} (x - x_0)^{n+1} = 0$$

MVT is a particular case of LRT, $n = 0$

$\sin(x)$, $x = 0.1$, $x_0 = 0$

$$\sin(x) = 0 + x + \cancel{\tfrac{0}{2} x^2} + \left(-\frac{\cos \xi}{3!}\right) x^3, \quad \xi \in (0, x)$$

$$|\sin(0.1) - 0.1| \leq \frac{1}{6} (0.1)^3$$

HW:

1. Let $F(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

   Show that $F \in C^\infty$ & $F^{(n)}(0) = 0$

2. $g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

   Show that $g'(x)$ exists $\forall x$
   but g' is not continuous

Pg. 192 # 23 a,b, # 24

April 4 2007

- Applications of Taylor's Theorem and of
- Lagrange Remainder formula

Locally f behaves as its Taylor Polynomial.

f defined in neighborhood of x_0, $f'(x_0)=0$ $\quad f \in C^n(\alpha,\beta)$ $\quad x_0 \in (\alpha,\beta)$

case 1: $n=2$ $\quad$ a) $f''(x_0)>0$ so x_0 is a local min

b) $f''(x_0)<0$ so x_0 is a local max

c) $f''(x_0)=0$ further investigation is necessary

$$f(x) = f(x_0) + \underset{\overset{\parallel}{0}}{f'(x_0)}(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + r(x) \quad (\text{where } r(x) = \underset{x \to x_0}{o}((x-x_0)^2))$$

$$\lim_{x \to x_0} \frac{|r(x)|}{|x-x_0|^2} = 0 \;. \quad \varepsilon = \left|\frac{f''(x_0)}{4}\right| \quad \exists\, \delta > 0 \text{ s.t. } \forall x \in (\alpha,\beta)$$

$$|x-x_0| < \delta \Rightarrow \frac{|r(x)|}{|x-x_0|^2} < \frac{|f''(x_0)|}{4}$$

---

estimate $f(x)$ for $f''(x_0)>0$

$$f(x) \geq f(x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 - |r(x)|$$

$$f(x) \geq f(x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 - \left|\frac{f''(x_0)}{4}\right|(x-x_0$$

$$f(x) \geq f(x_0) + \frac{f''(x_0)}{4}(x-x_1)^2$$

and can say

$$f(x) < f(x_0) + \frac{f''(x_0)}{4}(x-x_0)^2 \text{ if } x \neq x_0$$

$|x-x_0| < \delta$

$f''(x_0)<0$ $\quad |f''(x_0)| = -f''(x_0)$

$$f(x) \leq f(x_0) - \frac{|f''(x_0)|}{2}(x-x_0)^2 + |r(x)|$$

$$f(x) \leq f(x_0) - \frac{|f''(x_0)|}{4}(x-x_0)^2 < f(x_0)$$

$\nwarrow$ if $x \neq x_0$ and $|x-x_0| < \delta$

Now consider $f \in C^3$ and $f'(x_0) = f''(x_0) = 0$ and cases

a) $f'''(x_0) > 0$ } neither max nor min
b) $f'''(x_0) < 0$ }
c) $f'''(x_0) = 0$ further investigation is needed

$$f(x) = f(x_0) + \frac{f'''(x_0)}{6}(x-x_0)^3 + r(x) \quad \left(\text{where } r(x) = o((x-x_0)^3)\right)$$

local behavior of f is defined by behavior of taylor polynomial with nonzero highest order term.

Lagrange Remainder Formula. Convergence of Taylor's series

$f(x) = \sin(x)$	0
$f'(x) = \cos(x)$	1
$f''(x) = -\sin(x)$	0
$f'''(x) = -\cos(x)$	-1
$f''''(x) = \sin(x)$	0

$x_0 = 0$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{(2k+1)} = \sin(x)$$

prove this by estimating remainder ;

$$\left| \sin(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{(2k+1)}}{(2k+1)!} \right| = \left| \frac{f^{2n+2}(z)}{(2n+2)!} \right| |x|^{2n+2} \le$$

$$\frac{|x|^{2n+2}}{(2n+2)!} \longrightarrow 0 \quad (n \to \infty)$$

Order of Zeroes

$f(x_0)=0 \qquad x_0 \in (a,b),\ f \in C^N(a,b)$

$f'(x_0)$

$f''(x_0)$

$\vdots$

Order of zero at $x_0 = \min\{n : f^{(n)}(x_0) \neq 0\} - 1$

$= \max\{k : f^{(j)}(x_0) = 0 \ \forall j \leq k\}$

Apply Taylor's thm:

$$f(x) = \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k - o(x-x_0)^k$$

where k is order of zeroes at x_0.

* Zeroes of infinite orders exists!

Ex:

$\exists ? \quad f \in C^n(\mathbb{R})$ s.t. $\begin{cases} f(x)=0 & x \leq 0 \\ f(x) > 0 & x > 0 \end{cases}$

Can we construct such f?

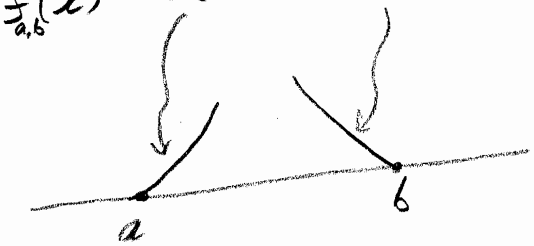
$$f(x) = \begin{cases} 0 & x < 0 \\ x^{n+1} & x > 0 \end{cases}$$

$\exists f \in C^\infty$ s.t. $\begin{cases} f(x)=0 & x \leq 0 \\ f(x)>0 & x>0 \end{cases}$

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x^2}} & x>0 \end{cases}$$

Find $f_{a,b} \in C^\infty(\mathbb{R})$ s.t. $\begin{cases} f(x)=0 & x\notin(a,b) \\ f(x)>0 & x\in(a,b) \end{cases}$

$f_{a,b}(x) = f(x-a)\cdot f(b-x)$ where: $f(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x^2}} & x>0 \end{cases}$



& the multiplication:

Corr:
Let $\underset{\text{closed}}{A} \subset \mathbb{R}$

$\exists f \in C^\infty(\mathbb{R})$
s.t. $\{x : f(x)=0\} = A$

Remark:
If $f \in C(\mathbb{R}) \Rightarrow \{x : f(x)=0\}$ is closed.

$f(x) = \text{dist}(x, A)$ - continuous function s.t $\{x : f(x)=0\} = A$

Thm:
Any open set B can be represented as a disjoint union of intervals (a_k, b_k)

Proof in the book.

$f(x) = \sum \alpha_k f_{a_k b_k}(x)$ with apropiate $\alpha_k > 0$

Lemma:

$f \in C(D)$ D - open.

$f \in C^1(D \setminus \{x_0\})$

If $\lim_{x \to x_0} f'(x)$ exists

then $f \in C^1(D)$ & $f'(x_0) = \lim_{x \to x_0} f'(x)$

Proof: use MVT

Apply L by induction!

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(x) = e^{-1/x^2} \cdot 2x^{-3}$$

need to show:

$$\lim_{x \to 0} 2x^{-3} \cdot e^{-1/x^2} = 0$$

$y = \frac{1}{x^2}$

$$\lim_{y \to \infty} \frac{2 \cdot y^{3/2}}{e^y} = 0$$

$\Downarrow$

$f \in C^1$

Apply L to $f' \dots f^{(n)}$

$$\left(e^{-\frac{1}{x^2}}\right)^{(n)} = P(x^{-1}) \cdot e^{-1/x^2}$$

(Using Lhópitalle's rule)

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

If $\begin{cases} \lim f(x) = \lim g(x) = \begin{cases} 0 \\ \infty \end{cases} \\ \& \\ \lim \frac{f'}{g'} \text{ exists} \end{cases}$

$\Downarrow$

e^x beats x^p beats $(\ln x)^q$ (at 0 & ∞)

Proof of Lhop.

2nd MVT:

$f, g \in C([a,b]) \cap C^1((a,b))$

then:

$\exists x_0 \in (a,b)$ s.t.

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$F(x) = (f(b)-f(a)) \cdot (g(x)-g(a)) - (g(b)-g(a)) \cdot (f(x)-f(a))$$

Apply Rolle's thm.

And get Lhop. for the case of $\frac{0}{0}$. (HW)

HW

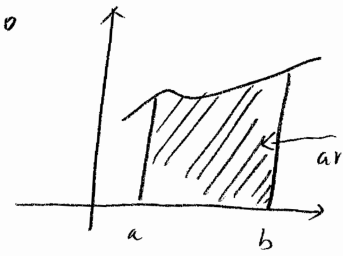
P. 194 #19, 20

4-9-07

Note by Jin H.

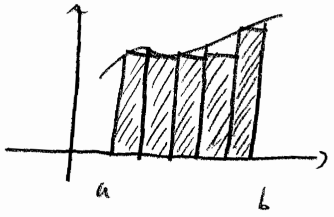
Riemann Integral

$f \geq 0$



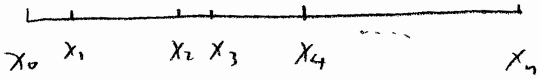
area $\int_a^b f(x)dx$

How to approximate:



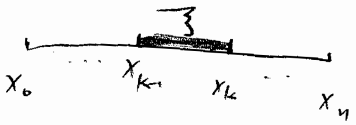
Def: A partision P of $[a,b]$ is $\{x_k\}_0^n$

s.t. $a = x_0 < x_1 < x_2 < \cdots < x_n = b$



partisions do not have to be equal.

- Rank P = $\max_k (x_k - x_{k-1})$
- Rigging of P, ξ_k, $k = 1, 2 \ldots n$, $\xi \in [x_{k-1}, x_k]$



∴ Riemann Sum:

$$s(f, P, \{\xi_k\}) = \Sigma f(\xi_k)\cdot(x_k - x_{k-1})$$

Also also Cauchy Sum

Def: f be boulded on $[a, b]$

$$\int_a^b f(x)\,dx = \underset{\substack{\text{rank } P \to 0 \\ \uparrow \\ \text{exist}}}{\lim} S(f, P, \{\xi_k\})$$

$$A = \lim_{P \to 0} S(f, P, \{\xi_k\}) \quad \text{if}$$

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \text{s.t. } \forall\, P,\ \{\xi_k\}$$

$$P < \delta \Rightarrow |S(f, P, \{\xi_k\} - A| < \varepsilon$$

Equivalently:

$$A = \lim_{P \to 0} S(f, P, \{\xi_k\}) \quad \text{if } \forall \text{seq } P_n,\ \text{s.t. } P_n \to 0$$

$$A = \lim_{P \to 0} S(f, P_n, \{\xi_k^n\})$$

Def f, P

$$S^+(f, P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

$$S^-(f, P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

where $M_k = \sup \{ f(x),\ x \in [x_{k-1}, x_k] \}$

$m_k = \inf \{ f(x),\ x \in [x_{k-1}, x_k] \}$

$$\underset{\substack{\uparrow \\ \text{lower}}}{S^-(f, P)} \leq S(f, P, \{\xi_k\}) \leq \underset{\substack{\uparrow \\ \text{upper Riemann Sum}}}{S^+(f, P)}$$

Thm let f be bounded on $[a, b]$

f is Remann integrable on $[a,b]$ iff:

$$\lim_{P\to 0} \overbrace{\left[S^{+}(f,P) - S^{-}(f,P)\right]}^{Osc(f,P)} = 0$$

integrable means:

$$\lim_{P\to 0} S(f, P, \{\xi_k\}) \text{ exist.}$$

Moreover, if f is Riemann integrable, then

$$\int_a^b f(x)\,dx = \sup_P S^{-}(f,P) = \inf_P S^{+}(f,P)$$

Pf: $\lim_{P\to 0} Osc(f, P) \to 0 \Rightarrow f$ is R. integrable

we know this:

$S^{-}(f,P) \le S^{+}(f,P)$

we want this:

$\Rightarrow \sup S^{-}(f,P) \le \inf S^{+}(f,P)$

*e.g.

$f > g$

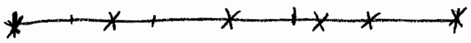
but $\sup g > \inf f$

$S^{-}(f, P_1) \le S^{+}(f, P_2)$ $(*)$

$P_3 = P_1 \cup P_2$

P_1 by $|$

P_2 by x



$\therefore$ P_3 by $|$ & x

$\therefore$ P_3 is refinement of P_1 & P_2

$$S^{-}(f, P_1) \le S^{-}(f, P_3) \le S^{+}(f, P_3) \le S^{+}(f, P_2)$$

$\therefore$ $(*)$ is true.

$\forall P_2$, $S^{-}(f, P_1) \le S^{+}(f, P_2)$

$\Rightarrow \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\sup S^{-}(f,P) \le \inf S^{+}(f, P_2)}$

Take arbitrary seq. of partisons P_k, s.t.

P_{k+1} is a refinement of P_k, rank $P_k \to 0$

$S^-(f, P_k) \nearrow$ · both are bounded monotonic functions

$S^+(f, P_k) \searrow$ $\Rightarrow$ lim exists.

$\therefore \lim S^+(f, P_k) - \lim S^-(f, P_k)$ exist

$\lim S^+(f, P_k) - \lim S^-(f, P_k) = 0$

Furthermore:

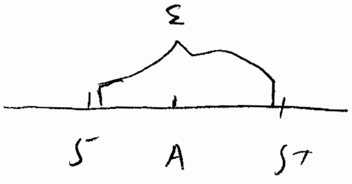
$\sup S^-(f, P) < \inf S^+(f, P)$ is impossible

$\lim S^-(f, P_k) \geq \underline{\inf S^+(f, P) \geq \sup S^-(f, P_k) \geq \lim_k} S^-(f, P_k)$

$\sup S^-(f, P) = \inf S^+(f, P) = A$

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $P < \delta$

$\Rightarrow |S^+(f, P) - S^-(f, P)| < \varepsilon$

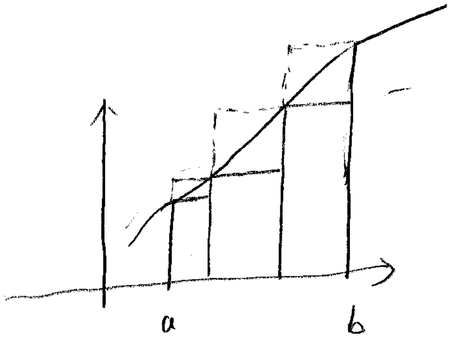


$\Rightarrow S^+(f, P), S^-(f, P) \in B_{A, \varepsilon} = (A - \varepsilon, A + \varepsilon)$

$\Rightarrow S(f, P, \{\xi_k\}) \in [S^-(f, P), S^+(f, P)]$

One immediate corrolary:

Bounded monotonic function is Riemann integrable.



$$s^+(f,P) - s^-(f,P) \leq |f(b)-f(a)| \cdot \text{Rank } P$$

HW: p 217 #2

p 231 # 1, 3, 4, 6

Kailin Clarke
4/11/07
MA101

At end of last class

$$\boxed{\begin{array}{l} f \text{ is R. Integrable} \\ \quad \text{iff } \operatorname{Osc}(f, P) \to 0 \text{ as Rank } P \to 0 \end{array}}$$

Remark If f is R. Integrable on $[a,b]$
then $\int_a^b f(x)\,dx = \lim S(f, P_n, \{\xi_k^n\})$
For any P_n s.t. rank $P_n \to 0$.

Remark
f is integrable on $[a,b]$
$\Rightarrow f$ is bdd

Thm f cont. on $[a,b]$
$\Rightarrow f$ is R, integrable on $[a,b]$

Cauchy did this:

And said "areas of rectangles above and below curve must have total sum of zero as rank $P_n \to 0$." Good idea, but Cauchy was wrong!

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{n}{n+k} = \lim_{k\to\infty} (1) = 1$$

$$\lim_{n\to\infty} \lim_{k\to\infty} \frac{n}{n+k} = \lim_{n\to\infty} (0) = 0$$

You cannot change the order of limits without justification!

Proof (and this one is CORRECT)

f is cont. on compact $[a,b]$ so f is uniformly cont. on $[a,b]$.

Fix arb $(\forall)$ $\varepsilon > 0$. $\exists\ \delta$ s.t. $\forall x, x' \in [a,b]$ $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \frac{\varepsilon}{b-a}$

Take a partition P, rank $P < \delta$

$[x_{k-1}, x_k]$ $\quad M_k - m_k = f(x^k_{max}) - f(x^k_{min})$

$x_k - x_{k-1} < \delta \Rightarrow |x^k_{max} - x^k_{min}| < \delta$

$$\underbrace{|M_k - m_k|}_{\geq 0} < \frac{\varepsilon}{b-a}$$

$$0 \le \overset{\geq 0}{M_k} - m_k < \frac{\varepsilon}{b-a}$$

$$\text{Osc}(f,P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \sum_{k=1}^{n} \frac{\varepsilon}{b-a} (x_k - x_{k-1})$$

$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

So $\text{Osc}(f,p)$ goes to zero and f is R. integrable

Ex $\mathcal{D}(x) = \begin{cases} 1 & x\text{ - rat} \\ 0 & x\text{ - irrat} \end{cases}$ not R. integrable at any interval because $\text{Osc}(\mathcal{D}, P) = b-a \quad \forall$ Part P

Thm f is R. integrable on $[a,b]$ if $\exists$ seq. of partitions P_n s.t. $\text{Osc}(f, P_n) \to 0$ as $n \to \infty$

Lemma (L 6.2.1) Let P, P'

rank $P' < \min$ of $\{x_k - x_{k-1} : k = 1, 2, \dots n\}$

$P = \{x_1, x_2, \dots, x_n\}$

Then $\forall f \quad \text{Osc}(f, P') < 3 \cdot \text{Osc}(f, P)$ (Look in book for proof)

Some elementary properties of integral

Additivity of integral

$a < b < c$

f integr on $[a,b]$ on $[b,c]$

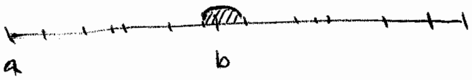
$\Rightarrow f$ int on $[a,c]$ & $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

If we know that f is integrable on $[a,c]$ then everything is trivial (and easy)

Pick P_n^1 on $[a,b]$ $\quad P_n^1 \cup P_n^2$ - part on $[a,c]$

P_n^2 on $[b,c]$

$\text{Rank } P_n^1 \to 0$
$\text{Rank } P_n^2 \to 0$ $\Longrightarrow \text{Rank } P_n = 0$



one idea would be to do it this way - merge the two partitions and figure out the additional contribution of the new partition. ~~But this is~~ But why do this when the above theorem gives it to you for free? Using the theorem is a better idea.

the one about a sequence of partitions

Linearity of integral

Fundamental theorem of calculus.

Let f cont on $[a,b]$

For $x \in [a,b]$ define

$F(x) = \int_a^x f(t)\,dt$ because if f is cont. on $[a,b]$ it is cont. on any smaller integral inside.

Then $F'(x) = f(x)$

Homework

$$\frac{d}{dx}$$

$$\int_0^{x^3} e^{-t^2} dt$$

not collected

$$\int_{x^2}^{x^3} e^{-t^6} dt$$

There will be a 2^{nd} midterm before the final

Take home part next friday → monday. In class part - monday in class

Corollary of FTC.

f continuous on $[a,b] \Rightarrow$ f has antiderivative ($\exists$ F s.t. $F'(x) = f(x)$ $\forall x$)

Antiderivative = primative
called

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)$$

antiderivative unique up to a constant.

We defined Riemann Integral $\int_a^b f(x)dx$ $a \leq b$

Define $\int_b^a f(x)dx := -\int_a^b f(x)dx$

can get this from

$\int f(x)\,dx$ = set of all antiderivatives of f

$= F(x)+C$

indefinite integral

change of variables. $\int f(u)du = F(u)$

then $\int f(g(x))\,g'(x)dx = F(g(x))$

need that f - continuous, $g \in C^1$

has continuous first derivative

Pf of $\uparrow$

Chain rule

$F(g(x))' = f(g(x)) \cdot g'(x)$

$\int_a^b f(g(x))\, g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

do not need to assume that $g \nearrow$

$\int u dv = uv - \int v du \qquad u, v \in C^1$

$dv(x) = v'(x)dx$

Improper integrals

R. Int. defined for finite intervals and bounded fxns.

$\int_0^1 \frac{1}{\sqrt{x}} dx$ goes to ∞ at 0. $\Rightarrow$ not $\int$

So can do $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}} dx$

Story of e

$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$

$a^x \quad a > 0$

naive way approximating usually — $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

$(a^x)' = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$. hard to show that this limit exists with

e – value of a s.t. $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$

$(e^x)' = e^x$

$\ln x$ – inverse to e^x. $e^{\ln x} = x$

from inverse fxn thm. — $(\ln x)' = \frac{1}{x}$

One idea is to define $\ln x$ as antiderivative of $\frac{1}{x}$

define $\ell(x) := \int_1^x \frac{1}{t}\,dt$ well defined for $x>0$

Make ℓ like $\ln$ $\quad \ell(1)=0 \quad (\ln(1)=0)$

$\ell'(x)=\frac{1}{x}$

$\ln(ab)=\ln a+\ln b$

$$\ell(ab)=\int_1^{ab}\frac{1}{t}\,dt$$

$$=\underbrace{\int_1^{a}\frac{1}{t}\,dt}_{\ell(a)}+\int_a^{ab}\frac{dt}{t}$$

$x=\frac{t}{a}$

$t=ax$

$dt=a\,dx$

$$\Rightarrow\quad =\ell(a)+\int_1^{b}\frac{\cancel{a}\,dx}{\cancel{a}x}=\ell(a)+\ell(b)$$

$$\ell(ab)=\ell(a)+\ell(b)$$

$\ell(x)\nearrow$ strictly $\quad \ell'(x)>0$

$\ell((0,\infty))=\mathbb{R}$

$\ell(2)>0 \quad \int_1^2 \frac{1}{x}\,dx>0$ its integral

$\ell(2\cdot 2)=2\ln(2)$

$n\in\mathbb{N}$ enough

$\ell(2^n)=n\,\ell(2)\to\infty$ as $n\to\infty$ $\quad (\ell(2)>0)$

$\Rightarrow$ ℓ not bounded above

$\ell(\frac{1}{2})<0 \quad \int_1^{\frac{1}{2}}\frac{1}{x}\,dx=-\int_{\frac{1}{2}}^{1}\frac{1}{x}\,dx<0$ — positive function $\Rightarrow$ integral of it >0

$\ell((\frac{1}{2})^n)=n\,\ell(\frac{1}{2})\to-\infty$ as $n\to\infty$

$\Rightarrow$ ℓ takes all $\mathbb{R}$.

$$\left[\overset{-\infty \nwarrow}{\ln(\tfrac{1}{2})}, \overset{\nearrow +\infty}{\ln(2)}\right]$$

by intermediate value theorem, it takes all values inbetween

$\Rightarrow$ takes all $\mathbb{R}$

Define $\exp(x)$ as inverse function to $l(x)$

$\Rightarrow$

$$\exp(l(x)) = x \quad \forall x \in (0,\infty)$$

$$l(\exp(y)) = y \quad \forall y \in \mathbb{R}$$

Use inverse function theorem

$$f : [a,b] \longrightarrow [f(a), f(b)]$$

monotone, C^1, bijection

$g = f^{-1}$

$$g(f(x)) = x \quad f(g(x)) = x$$

$\forall x \in [a,b]$ $\quad$ $\forall x \in [f(a), f(b)]$

If $f'(x) \neq 0 \Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}$

Hard part is to show that $g'(f(x))$ exists

If you do know it exists, you can use chain rule

$$g(f(x)) = x$$

$$g'(f(x))\, f'(x) = 1$$

divide by $g'(x) \rightarrow$ get formula

$\Rightarrow$ $\exp(x)' = \exp(x)$

$\exp(0) = 1$

($l(a) + l(b)$ gives $e^{x+y} = e^x e^y$)

Homework

p. 218 #5

p. 232 #9

p. 235 #3

p. 335 #6,7

April 16 MA0101 Sheet 1

Note
taker

$\ell(x) = \int_1^x \frac{1}{t}\,dt$

$\ell'(x) = \frac{1}{x}$

$\ell(x) = \ln(x)$

$epx(x) = e^x$

exp = inverse of $\ell(x)$

$\ell(\exp(x))' = x'$ → derivative → $\frac{1}{\exp(x)} \cdot \exp(x)' = 1$

$\exp(x)' = \exp(x)$

$$\boxed{\lim \left(1+\frac{1}{n}\right)^n = e}$$

$$\begin{cases} y'(x) = y \\ y(0) = 1 \end{cases}$$

$y(x) = e^x$ is a solution

unique solution

Simplest numerical method to solve differential equation is Eulers' method:

$y' = f(x,y)$

$y_{n+1} = y_n + f(x_n, y_n) \cdot h$

$[0, a]$ n parts $\frac{a}{n}$

$x_0 = 0$

$y_0 = 1$

$$y(x_1) \approx y'(x_0) \cdot \Delta X + y_0 = 1 + a_n = y_1$$

$$y_2 = \underbrace{\left(1+\frac{a}{n}\right)}_{y_1} + \underbrace{\left(1+\frac{a}{n}\right)}_{f(x_1 y_1)} \cdot \frac{a}{n} = \left(1+\frac{a}{n}\right)^2 \approx y(x_2)$$

$$y_n = \left(1+\frac{a}{n}\right)^n \approx y(x_n) = y(a) = e^a$$

Eulers' method converges

So $\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a$

Cauchy criterion of convergence of Σa_n since

$\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon > 0$ $\exists N$ s.t

$\forall$ $m \geq n > N$ $\underbrace{\left| \sum_{l=n}^{m} a_k \right|}_{S_m - S_{n-1}} < \varepsilon$

Necessary condition of convergence

Thm Σa_n converges

$\Rightarrow \lim_{n \to \infty} a_n = 0$

Attn, converse not true!!!

Particular case $n=m$ of Cauchy test gives exacly def of $\lim a_n = 0$

Absolute Convergence

Σa_n absolute convergent if $\Sigma |a_n|$ converges

Thm $\Sigma |a_n|$ converges $\Rightarrow$ Σa_n converges

pf: $\Sigma |a_n|$ converges $\Rightarrow$

$\forall \varepsilon > 0$ $\exists N$ s.t $\forall$ $m \geq n > N$

$\sum_{k=1}^{m} |a_k| < \varepsilon$

Sheet 3

Prop If $A = \mathbb{N}$ $\quad a_x \geq 0$

then $\displaystyle \sup_{\substack{\sigma \subset \mathbb{N} \\ \text{finite}}} S_\sigma = \lim_{n \to \infty} S_n$

Pf. $a_n \geq 0 \Rightarrow S_n \nearrow$

so $\displaystyle \lim_{n\to\infty} S_n = \sup_n S_n$

$$\Big(\text{Trivially } \sup_{\sigma} \underset{\substack{\uparrow \\ \text{all possible} \\ \text{finite sums}}}{S_\sigma} \geq \sup \underset{\substack{\nwarrow \text{ only specific} \\ \text{finite sums (from 1 to n)}}}{S_n}\Big)$$

Take any finite $\sigma \subset \mathbb{N}$

$\exists\, n$ s.t. $\sigma \subset \{1, 2, \ldots, n\} \Rightarrow S_\sigma \leq S_n$

$\Rightarrow \sup S_\sigma \leq \sup S_n$

$\therefore \sup S_\sigma = \sup S_n$

HW
§7.2
p262 #1, 2, 3, 9, 13, 14

MA 101 4/18/07

$$\sum_{\alpha\in A} a_\alpha = \sup_{\sigma\text{-finite}} S_\sigma \qquad \text{where } S_\sigma = \sum_{\alpha\in\sigma} a_\alpha$$

$a_\alpha \geq 0$

Remark: If $a_\alpha \geq 0 \ \forall \alpha \in A$, & $\sum_{\alpha\in A} a_\alpha = S < \infty$,

then $\{\alpha : a_\alpha \neq 0\}$ is either finite or countable.

Pf: Fix $n \in \mathbb{N}$

$$\text{Card}\{\alpha : a_\alpha \geq \tfrac{1}{n}\} \leq S \cdot n$$

$$\{\alpha : a_\alpha > 0\} = \bigcup_{n=1}^{\infty} \underbrace{\{\alpha : a_\alpha \geq \tfrac{1}{n}\}}_{\text{Each set is finite}}$$

Infinite union of finite sets is either finite or countable.

Reordering does not change the sum of $\sum a_n$, $a_n \geq 0$

Absolutely convergent series:

$a_n^+ = \max\{a_n, 0\}$

$a_n^- = \max\{-a_n, 0\}$

$a_n = a_n^+ - a_n^-$

Example: If $a_n = 1$, $a_n^+ = 1$ $a_n^- = 0$

If $a_n = -1$ $a_n^+ = 0$ $a_n^- = -1$

Thm: $\sum |a_n|$ converges iff

$\sum a_n^+$ converges & $\sum a_n^-$ converges.

Pf: If $\sum a_n^+, \sum a_n^-$ converge, Then $\underset{\uparrow \text{finite}}{\sum |a_n|} = \sum (a_n^+ + a_n^-)$

Pf of converse: If $\sum |a_n|$ converges, $a_n^+, a_n^- \leq |a_n|$ $a_n^+, a_n^- \geq 0$

By the comparison test, $\sum a_n^+$ and $\sum a_n^-$ converge.

If $\sum |a_n| < \infty$

then $\sum a_n = \sum a_n^+ - \sum a_n^-$

All terms positive, difference does not depend on renumbering.

Cor: If $\sum a_n$ converges absolutely,

$\Rightarrow \sum a_n$ does not depend on renumbering.

Renumbering: $\{a_k\}$ $\varphi : N \to \mathbb{N}$ bijection.

$$\{a_{\varphi(n)}\}_{n=1}^{\infty}$$

normal: $a_1 \quad a_2 \quad a_3 \quad a_4 \dots$

renumbering: $\underset{\varphi(1)}{\underset{\shortparallel}{a_{21}}} \quad \underset{\varphi(2)}{\underset{\shortparallel}{a_{17}}} \quad \underset{\varphi(3)}{\underset{\shortparallel}{a_1}} \quad a_4$

Thm: Let $\sum a_n$ converges, but not absolutely., $a_n \in \mathbb{R}$

$(\sum |a_n| = \infty)$

Then $\forall x \in \mathbb{R} \ \exists \underset{\text{bijection}}{\varphi : \mathbb{N} \to \mathbb{N}}$ s.t. $\sum_{n=1}^{\infty} a_{\varphi(n)} = x$

Tests of convergence:

1. Comparison Test

$$0 \le a_n \le b_n$$

If $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

$$\Updownarrow$$

If $\sum a_n > \infty \Rightarrow \sum b_n > \infty$

Remark: ~~If $\sum b_n = \infty \Rightarrow \sum a_n = \infty$~~ faulty logic

If $|a_n| \le b_n \; \forall n$

$\Rightarrow \sum a_n$ converges absolutely

---

2. Ratio Test

Let $\lim\limits_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = r$

If $r < 1$, series converges absolutely

If $r > 1$, series diverges, $\lim\limits_{n \to \infty} a_n \neq 0$

If $r = 1$, anything can happen

---

3. Root Test

$\limsup\limits_{n \to \infty} |a_n|^{1/n} = r$

If $r < 1$, series converges absolutely

If $r > 1$, series diverges $\lim\limits_{n \to \infty} a_n \neq 0$

If $r = 1$, anything can happen

Proofs for 2 and 3 found by comparison with geometric series.

4. Alt. series Test

$a_n \searrow 0$ $(a_{n+1} \leq a_n, \lim_{n\to\infty} a_n = 0)$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n$ converges

---

5. Integral test:

Let $f: \mathbb{R} \to \mathbb{R}$ $f \downarrow$ $f \geq 0$

$$\sum_{n=1}^{\infty} f(n) < \infty \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

Main applications: p-series $(\sum \frac{1}{n^p})$

Ratio and root test, $r=1$

but $\int_1^{\infty} \frac{1}{x^p}\,dx$ is easy to find.

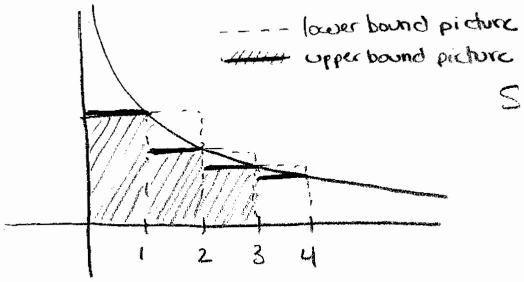
$\sum_1^{\infty} \frac{1}{n} = \infty$

Estimate N s.t. $\sum_{n=1}^{N} \frac{1}{n} \geq 1000$

HW: All HW from monday except problem 9

4/20

$\sum_{n=1}^{N} \frac{1}{n} \geq 1000$ for what N?



So $\sum_{n=1}^{N} \frac{1}{n} \geq \int_{1}^{N+1} \frac{1}{x}\, dx = \ln(N+1)$

is lower bound

$\sum_{n=1}^{N} \frac{1}{n} \leq 1 + \frac{1}{x}\, dx = 1 + \ln N$

is upper bound

This is a proof of the integral test, since one can use any decreasing function.

Comparison test can be proved using Cauchy criterion

Ratio and root tests are proved by comparison with geometric series : $\sum_{n=1}^{\infty} r^n$

$\sum_{n=0}^{\infty} x^n$ converges iff $|x| < 1$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Example $\sum_{k=3}^{\infty} \left(\frac{4}{5}\right)^{2k} = \sum_{k=3}^{\infty} \left(\left(\frac{4}{5}\right)^2\right)^k = \frac{(4/5)^6}{1-(4/5)^2}$

formally: $k = n+3$

$$\sum_{n=0}^{\infty} \left(\frac{16}{25}\right)^{3+n} = \left(\frac{16}{25}\right)^3 \sum_{n=0}^{\infty} \left(\frac{16}{25}\right)^n$$

Remark One can multiply absolutely convergent series

If $\sum |a_n| < \infty$, $\sum |b_n| < \infty$, then $\sum_{n,k=1}^{\infty} |a_n b_k| < \infty$

and $(\sum a_n)(\sum b_n) = \sum^{\infty} a_n b_k$

The interesting part is actually series of functions:

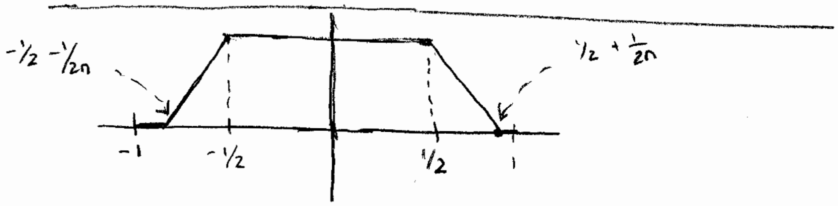
$$\sum_{n=1}^{\infty} f_n(x), \quad \text{specifically power series}: \sum_{n=0}^{\infty} a_n x^n$$

Uniform convergence

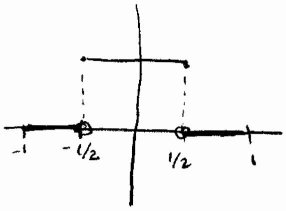
If $f_n \in C[a,b]$ & $\lim\limits_{n\to\infty} f_n(x) = f(x) \not\Rightarrow f \in C([a,b])$

$f_n(x) = x^n$ on $[0,1]$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$



$\lim\limits_{n\to\infty} f_n(x) =$

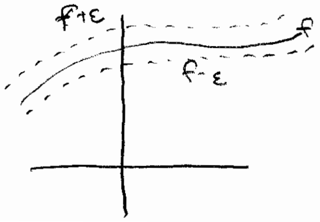


difference is underlined quantifier

Def f_n converge uniformly to f $(f_n \rightrightarrows f)$ $(f_n : \mathcal{D} \to \mathbb{R})$ if $\forall \varepsilon > 0\ \exists N\ \forall n > N\ \underline{\forall x \in \mathcal{D}}\ |f_n(x) - f(x)| < \varepsilon$

Remark $f_n(x) \to f(x)\ \forall x$ means $\underline{\forall x \in \mathcal{D}}$, $\forall \varepsilon > 0$ $\exists \varepsilon > 0$ $\exists N = N(\varepsilon, x)$ s.t. $\forall n > N$ $|f_n(x) - f(x)| < \varepsilon$

Uniform convergence means taking an ε-tube



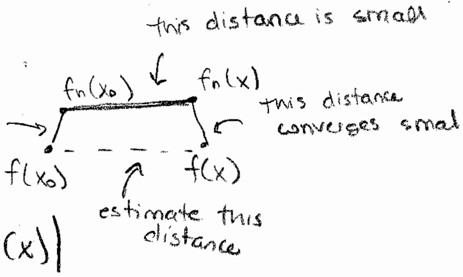
Theorem $f_n \in C(\mathcal{D})$ $f_n \rightrightarrows f$

Then $f \in C(\mathcal{D})$

Proof (by $\varepsilon/3$ trick)

Take $^{\forall} x_0 \in \mathcal{D}$, $^{\forall}\varepsilon > 0$

$|f(x_0) - f(x)|$



$$= |f(x_0) - f_n(x_0) + f_n(x_0) - f_n(x) + f_n(x) - f(x)|$$
$$\leq \underbrace{|f_n(x_0) - f(x_0)|}_{} + |f_n(x_0) - f_n(x)| + |f_n(x) - f(x)|$$

$f_n \rightrightarrows f \Rightarrow \exists N$ s.t. $\forall n > N$ $\forall x \in \mathcal{D}$ $|f_n(x) - f(x)| < \varepsilon/3$

Fix particular $n > N$

f_n -continuous $\Rightarrow \exists \delta > 0$ s.t. $\forall x \in \mathcal{D}$ $|x - x_0| < \delta$
$\Rightarrow |f_n(x) - f_n(x_0)| < \varepsilon/3$

Therefore $\forall x \in \mathcal{D}$ $|x - x_0| < \delta$
$$\Rightarrow |f(x) - f(x_0)| \leq |f_n(x_0) - f(x_0)| + |f_n(x) - f(x)| + |f_n(x_0) - f_n(x)|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$\Rightarrow f$ is continuous at x_0
(and since we don't assume anything about x_0, it's continuous everywhere)

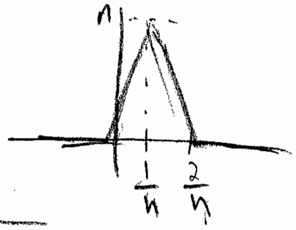
Remark f_n unif. cont. on $\mathcal{D}$, $f_n \rightrightarrows f$

$\Rightarrow f$- u. cont. on $\mathcal{D}$

Integration and Differentiation of Limits and Series.

1. $\int_a^b \left[\lim_{n\to\infty} f_n(x)\right] dx \neq \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

ex:



$\int_0^2 f_n(x)\,dx = 1 \ \forall n$

$\lim_{n\to\infty} f_n(x) = 0 \ \forall x$

---

2. $\int_a^b \left[\sum f_n(x)\right] dx \neq \sum \int_a^b f_n(x)\,dx$

Theorem: f_n - R.I. on $[a,b]$ and $f_n \rightrightarrows f$ on $[a,b]$

Then f is R.I. on $[a,b]$ and $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

(Partial) Proof for continous $f_n \Rightarrow f$ is then continuous also.

$f_n \rightrightarrows f \Rightarrow \forall \varepsilon > 0 \ \ \exists N$ s.t. $\forall n > N \ \ \forall x \in [a,b]$

$|f_n(x) - f(x)| < \varepsilon/(b-a)$

$$\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| = \left| \int_a^b (f(x) - f_n(x))\,dx \right| \le \int_a^b |f(x) - f_n(x)|\,dx < \varepsilon\cancel{(b-a)}$$

remark: a, b are finite.

---

3. $\lim f_n'(x) \neq (\lim f_n(x))'$

$f_n(x) = \frac{1}{n}\sin nx$ $\quad \lim f_n(x) = 0$, but $(\lim f_n(x))' \not\exists$.

4. Theorem: $f_n \in C^1[a,b]$, $f_n(x) \to f(x) \ \forall x \in [a,b]$, $f_n' \rightrightarrows g$ on $[a,b]$

Then $f \in C^1([a,b])$ and $f'(x) = g(x)$

Proof: fix $x_0 \in [a,b]$ $\quad f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'(t)\,dt$ so

$$f(x) = f(x_0) + \lim_{n\to\infty} \int_{x_0}^x f_n'(t)\,dt = f(x_0) + \int_{x_0}^x g(t)\,dt$$

the last step in the ploof was justified because $f_n' \rightrightarrows g$

By FTC $f'(x) = g(x)$

---

remarks: 1. It is also true that $f_n \rightrightarrows f$

2. Thm: $f_n \in C(a,b)$, ① $\exists x_0$ s.t. $\lim_{n\to\infty} f_n(x_0) = y_0$,

② $\forall \alpha, \beta \in (a,b)$ $f_n' \rightrightarrows g$ on (α,β)

Then $\exists f(x) = \lim_{n\to\infty} f_n(x)$ and $f'(x) = g(x)$ $\forall x \in (a,b)$

---

consider series:

$1 + x + x^2 \ldots x^n = \frac{1}{1-x}$ if $|x| < 1$

$\sum_{n=0}^{\infty} x^n$ converges uniformly on $|x| < r$ $\forall r < 1$

so $\sum_{n=1}^{\infty} n x^{n-1} = (1-x)^{-2}$

4/27/07

Def $\sum_{n=0}^{\infty} f_n(x)$ $\qquad f_n : D \to \mathbb{R}$

$\sum f_n$ converges uniformly means $S_n(x) = \sum_{k=0}^{n} f_k(x)$ converges unif. on $\mathcal{D}$.

Thm (sufficient condition of uniform convergence)

Let $a_n \geq 0$ st $\sum a_n < \infty$

suppose $|f_n(x)| \leq a_n$ $\forall x \in \mathcal{D}$. Then $\sum f_n(x)$ converges uniformly.

pf) $\forall x$ $\sum f_n(x)$ converges absolutely by comparison test.

$$|f_n(x)| \leq a_n \qquad S(x) = \sum_{n=0}^{\infty} f_n(x)$$

$$\sum a_n < \infty \Rightarrow \forall \varepsilon > 0 \ \exists N \text{ s.t. } \forall m > n > N \ \sum_{k=n+1}^{m} a_k < \varepsilon/2$$

$$|S_m(x) - S_n(x)| = \left|\sum_{k=n+1}^{m} f_k(x)\right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} a_k < \varepsilon/2.$$

$$\lim_{m\to\infty} |S_m(x) - S_n(x)| \leq \lim_{m\to\infty} \frac{\varepsilon}{2} \qquad |S(x) - S_n(x)| \leq \varepsilon/2 < \varepsilon \quad (Q.E.D)$$

Power Series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \ , \ \sum_0^{\infty} a_n x^n$$

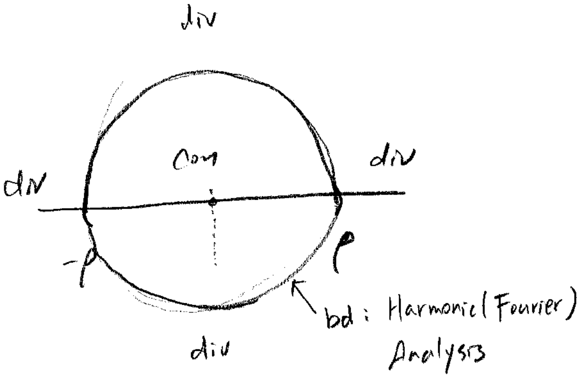
<complex field> $\sum a_n z^n$

Radius of Convergence

Thm $\sum a_n x^n$ $\exists! \rho \in [0, \infty]$ st.

1. $\forall x$ $|x| < \rho \Rightarrow \sum a_n x^n$ converges (absolutely)
2. $\forall x$ $|x| > \rho \Rightarrow \sum a_n x^n$ diverges

$\lim_{n\to\infty} a_n x^n \neq 0$



$\sum |a_n x^n|$

$\limsup |a_n x^n|^{1/n} = \limsup_{n\to\infty} |x| \cdot |a_n|^{1/n} = q$

abs. convergence if $q<1$.

div. ($|a_n x^n| \not\to 0$ $n\to\infty$) if $q>1$

$\rho = 1/(\limsup |a_n|^{1/n})$

$\rho = \infty$: conv. for all x $\rho=0$: Con. for only one x.

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\sum_{n=0}^{\infty} n!\, x^n$, $\sum_{n=0}^{\infty} n^n x^n$

Thm Let $\sum a_n x_n$, ρ - radius of conv. , $0<r<\rho$

Then $\sum a_n x^n$ conv. uniformly for $|x| \leq r$

pf) $\sum |a_n r^n| < \infty$ and $\forall x$ $|x| \leq r$

$|a_n x^n| \leq |a_n r^n|$

$\therefore \sum a_n x^n$ conv. uniformly on $|x| \leq r$

$\operatorname{Exp}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, prove using the multiplication of power series that

$E(x) \cdot E(y) = E(x+y)$ 〃

Power Series
4-30-07

Integration & differentiation of power series

① $\sum_{n=0}^{\infty} a_n x^n = f(x) \quad |x| < \rho \rightarrow$ radius of convergence

② $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} n a_{k+1} x^k$ ③ $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

Radii of convergence for series ② & ③ coincide with ρ

② $\limsup_{n\to\infty} (n|a_n|)^{1/n-1} = \limsup_{n\to\infty} (n|a_n|)^{1/n}$ (w/o changing radius of conv.) $\frac{1}{\rho} = \limsup_{n\to\infty} (|a_n|)^{1/n} \quad \sum a_n x^n$

$= \lim_{n\to\infty} n^{1/n} \cdot \limsup_{n\to\infty} |a_n|^{1/n} = 1 \cdot \limsup_{n\to\infty} |a_n| = 1/\rho$

$\hookrightarrow$ b/c limsup = maximal limit of a convergent subsequence; $\lim |n_k a_{n_k}|^{1/n_k} = \overset{=1}{\lim (n_k)^{1/n_k}} \cdot \lim |a_{n_k}|^{1/n_k}$

$\hookrightarrow \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln n^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^{\lim_{n\to\infty} \frac{1}{n}\ln n}$ (b/c e^x is cont.) $= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \underset{x\in\mathbb{R}}{=} e^{\lim_{x\to\infty} \frac{1}{x}}$ by L'Hospital $= e^0 = 1$

③ Same reasoning as for ②, but divide by n instead of multiply by n

- $f_n(x) = \sum_{k=0}^{n} a_n x^n \quad F_n(x) = \sum_{k=0}^{n} \frac{a_n}{n+1} x^{n+1}$

$F_n(x) = \int_0^x f_n(t)\, dt \quad F_n(0) = 0$

$\forall r < \rho \quad f_n \rightrightarrows f \quad |x| \le r$

$\forall x, |x| < \rho, \quad \underset{F_n(x) \xrightarrow[n\to\infty]{} F(x)}{\overset{\shortparallel}{\int_0^x f_n(t)\, dt}} \xrightarrow[n\to\infty]{} \int_0^x f(t)\, dt$

$\therefore F(x) = \int_0^x f(t)\, dt \quad \therefore F'(x) = f(x)$

- $f'(x) = g(x) \quad \forall x, |x| < \rho$

$g_n(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad f_n(x) = \int_0^x g_n(t)\, dt + a_0$

$g_n(t) \rightrightarrows g(t)$ on $[0, x]$

$\int_0^x g_n(t)\, dt \underset{n\to\infty}{\rightrightarrows} \int_0^x g(t)\, dt \quad f_n(x) \to f(x)$

$\therefore f'(x) = g(x) \quad |x| < \rho$

Term by term integration & differentiation doesn't change radius of convergence

- **Examples**

- Power series for $\ln(1-x)$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \int_0^x \frac{1}{1-t}\, dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \int_0^x \frac{1}{1-t}\, dt = -\ln(1-t)\Big]_0^x = -\ln(1-x)$

$\therefore \ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n \quad |x| < 1$

- Power series for $\tan^{-1}(x)$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \int_0^x \frac{1}{1+t^2}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} \quad \int_0^x \frac{1}{1+t^2}\, dt = \tan^{-1}(t)\Big]_0^x = \tan^{-1}(x)$

$\therefore \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

Mean for Exam 2:
64/100 66/105 (?)

Exam:
3 hrs. (More info to come later)
Cut off for A: ≈ 80% for the course

Instead of homework, review material & write down important facts.