9/6/07

Equivalence Relations

Quotient Space

Let X be a set and $\sim$ be an equivalence relation on X

Take $x \in X$ → define the corresponding coset

$$[x] = \{y \in X : y \sim x\}$$

If $[x], [y]$, then there are only 2 possibilities:

1. $[x] \cap [y] = \emptyset$
2. $[x] = [y]$

All cosets are disjoint

PF If $[x] \cap [y] \neq \emptyset$

$\Rightarrow \exists z \in X$ s.t. $z \in [x], z \in [y]$

$z \sim x \quad z \sim y$

$\Rightarrow x \sim y$ by transitivity

Then if $w \in [x] \Rightarrow w \sim x$

$\underset{x \sim y}{\Rightarrow} w \sim y \Rightarrow w \in [y]$

and if $w \in [y] \Rightarrow w \in [x]$

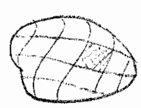
$\Rightarrow [x] = [y]$

Example:

$\mathbb{Q}$ - rational numbers $\qquad p/q$ - fraction

$p/q \sim r/s$ iff $ps = rq$

The collection of all cosets is a quotient space $X/\sim$



All elements of a coset are representations of the same object

Example continued: $\qquad$ different fractions represent one rational #

$\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}\}/\sim$

Example:

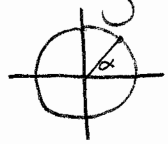
$\mathbb{R}$

$x \sim y \Leftrightarrow x - y \in \mathbb{Z}$

$\mathbb{R}/\sim$ is a circle

$\mathbb{R}/\mathbb{Z}$

Position on a circle is determined by an angle



$x \cdot 2\pi \leftrightarrow$ angle

A relation is a set

Formally, $(x, y) \in R$

Relation notation: xRy

Ex ~~$2, 3 \in <$~~

$2 < 3$

Partial Order → read about this in the textbook.

<u>Function</u>

Informal definition: $f: X \to Y$ is a triple:

1. X - domain
2. Y - target space, codomain
3. "rule" f that assigns to <u>each</u> $x \in X$ an <u>unique</u>! element $y \in f(x) \in Y$

More formal definition: $f: X \to Y$ is a triple:

1. X - domain
2. Y - codomain
3. f - a relation on $X \times Y$ s.t. $\forall x \in X \ \exists! y \in Y$ s.t. $\underline{xfy}$

$y = f(x) \leftarrow$ function notation

<u>Def</u> $f: X \to Y$ is called

1. Injective if $\forall x_1, x_2 \in X \quad x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$
   ↳ 1-1, into
2. Surjective if $\forall y \in Y \ \exists x \in X$ s.t. $y = f(x)$
   ↳ onto
3. Bijective = injective AND surjective

Examples:

1. $f: \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$

   not injective $\to f(-x) = f(x)$

   not surjective $\to \nexists x$ s.t. $f(x) = -1$

2. $f : \underbrace{[0,\infty)}_{\mathbb{R}_+} \to \mathbb{R}$ $\quad f(x) = x^2$

injective, not surjective

3. $f : \mathbb{R} \to \mathbb{R}_+$ $\quad f(x) = x^2$

surjective, not injective

4. $f : \mathbb{R}_+ \to \mathbb{R}_+$ $\quad f(x) = x^2$

bijective

f is a function

$f(x)$ is the value (of f) at x

Technically, need to consider $x \mapsto x^2 + \sin x$ rather than $x^2 + \sin x$ and to define the domain and target space

$f : X \to Y$

$\underbrace{f(X)}$ $= \{ y \in Y : \exists x \in X \ f(x) = y \}$

↳ range or image of f (Ran f or Im f)

Homework: Sections 1.1, 1.3

Read 1.2

Page 23 #2, 14, 15

In example 1

Ran $f = \mathbb{R}_+$