Homework Review: 9/10

$(0,1) \to \mathbb{R}$

$$\frac{x^2+x-1}{x^3-x} \quad \text{or} \quad \tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

Check?

$f : (0,1) \to \mathbb{R}$

$f(x) = \tan(\pi x - \pi/2)$

$(0,1) \to [0,1]$

$\left(\frac{1}{n+1}, \frac{1}{n}\right) \to \left(\frac{1}{n+1}, \frac{1}{n}\right)$

$\frac{1}{n}, \ n \in \mathbb{N} \qquad \frac{1}{n}, \ n \in \mathbb{N}$

Construct bijection

$\{\frac{1}{n} : n \in \mathbb{N}\} \to \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \{1\}$

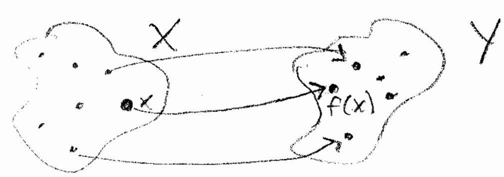
$f(x) = x \quad x \neq \frac{1}{n}$

Visualization of functions

- Graph

function as relation is also a graph

- Drawing Arrows



Synonyms: function, map, mapping, transformation

Composition of functions

$f : X \to Y, \quad g : Y \to Z$

$g \circ f : X \to Z$

$g \circ f(x) = g(f(x))$

Inverse function

$f: X \to Y$ bijection

$\exists f^{-1}: Y \to X$ s.t. $f^{-1}(f(x)) = x \quad \forall x \in X$

$f(f^{-1}(y)) = y \quad \forall y \in Y$

f^{-1} is also a bijection

<u>Cardinality</u>

<u>Def.</u> X, Y $\text{Card } X = \text{card } Y$ If $\exists$ bijection $f: X \to Y$

$f^{-1}: Y \to X$ bijection

Card X = Card Y equivalence relation

$X \xleftrightarrow{f} \{1, 2, \dots, n\}$

Card $X = n$
X - finite

If Card X = Card $\mathbb{N}$, then we say X is countably infinite

$\exists f: \mathbb{N} \to X$ bijection

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finite or countably infinite sets are called countable (sometimes "countable is used to refer to only "countably infinite," but our book uses it to mean both)

$\exists$ uncountable sets.

Def. X, $\quad 2^X = \mathcal{P}(X)$ - set of all subsets of X

Thm. There is no surjective mapping $X \to 2^X$

Pf. Let $f: X \to 2^X$ surjection

Define $A = \{x \in X : x \notin f(x)\}$

f - surjective, so $\exists\, a \in X$ s.t. $A = f(a)$

2 possibilities:

1) $a \in A \Rightarrow a \notin f(a) = A$ $\Rightarrow \Leftarrow$
$\Rightarrow a \notin A$

2) $a \notin A \Rightarrow a \in f(a) = A$ $\Rightarrow \Leftarrow$

Cor. $\operatorname{Card} X \neq \operatorname{Card} 2^X$

$\operatorname{Card} \mathbb{N}$, $\quad \operatorname{Card} 2^{\mathbb{N}}$

And we can construct $2^{2^{\mathbb{N}}}$ (which has an even larger cardinality)

Is there anything between finite and $\operatorname{Card} \mathbb{N}$?

Thm If $X \subset \mathbb{N}$ is infinite, then $\operatorname{Card} X = \operatorname{Card} \mathbb{N}$

Pf Want: $f: \mathbb{N} \to X$

$f(1) = \min\{k : k \in X\}$

$f(2) = \min\{k : k \in X,\ k > f(1)\}$

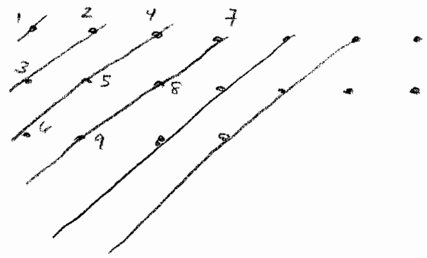
$\dots$

$f(n) = \min\{k : k \in X : k > f(n-1)\}$

Thm X is countable iff $\exists$ inj : $X \to \mathbb{N}$
iff $\exists$ surj : $\mathbb{N} \to X$

Thm $\operatorname{Card} \mathbb{N} \times \mathbb{N} = \operatorname{Card} \mathbb{N}$

Pf.



Cor. Countable union of countable sets is countable

So $\left.\begin{array}{l}\mathbb{Z}\\ \mathbb{Q}\end{array}\right\}$ countable