Math 113 Michael Foley 9/12/07

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Thm: X is countable (1)

iff $\exists$ inj: $X \to \mathbb{N}$ (2)

iff $\exists$ sur: $\mathbb{N} \to X$ (3)

(1) $\Rightarrow$ $\exists$ bijection $f: X \to \mathbb{N}$

$\Rightarrow$ f is injective

& f^{-1} is bijective, so surjective

$f^{-1}: \mathbb{N} \to X$

$\Leftarrow$ consider $X \to$ infinite $X \to$ finite is trivial

Thm is trivial for finite X

let $\exists$ inj $f: X \to \mathbb{N}$

then $f: X \to f(X) \subset \mathbb{N}$
bijection

X infinite, $f(X)$ infinite

$\Rightarrow f(X)$ inf. countable

$\Rightarrow \exists$ bijection $g: f(X) \to \mathbb{N}$ ($f(X)$ is countable)

then $g \circ f: X \to \mathbb{N}$ is a bijection

let $\exists$ surj. $f: \mathbb{N} \to X$

$\forall x \in X$ the set $f^{-1}(x) = \{n \in \mathbb{N} : f(n) = x\} \neq \emptyset$

$\forall$ set $f^{-1}(x)$ pick one element $n \in f^{-1}(x)$ and call it $g(x)$

then $g:$ ~~$\mathbb{N} \to X$~~ injection

$X \to \mathbb{N}$

so X ~ countable

$f^{-1}(x) \subset \mathbb{N}$ so it has minimal element and is not empty

define $g(x)$ = minimal element of $f^{-1}(x)$

Comparison of Cardinalities

Dfn: $\operatorname{card} A \leq \operatorname{card} B$ if $\exists$ injection $f: A \to B$ } this relation is not symmetric

Thm: (Cantor-Bernstein)

$\operatorname{Card} A \leq \operatorname{Card} B$

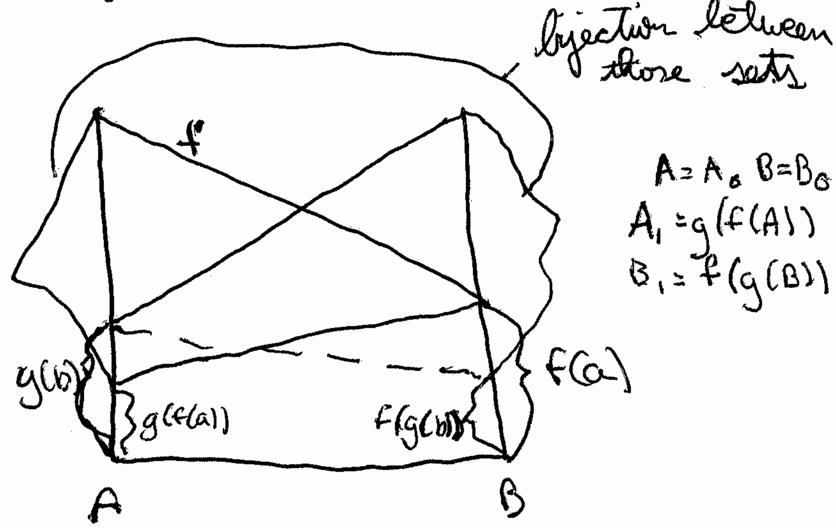
& $\operatorname{Card} B \leq \operatorname{Card} A$

$\Rightarrow \operatorname{Card} A = \operatorname{Card} B$

$\exists$ inj $f: A \to B$

$\exists$ inj $g: B \to A$

$\Rightarrow \exists$ bijection $\varphi: A \to B$ $\longrightarrow$



$A = A_0 \quad B = B_0$

$A_1 = g(f(A))$

$B_1 = f(g(B))$

- f defines bijection $A_0 \setminus g(B) \to f(A_0) \setminus \underbrace{f(g(B_0))}_{B_1}$
- g defines bijection $B_0 \setminus f(A_0) \to g(B_0) \setminus \underbrace{g(f(A_0))}_{A_1}$
- g^{-1} bijection: $g(B_0) \setminus A_1 \to B_0 \setminus f(A_0)$

$f: A_0 \setminus A_1 \to B_0 \setminus B_1$

bijection

define $A_{n+1} = g(f(A_n))$

$B_{n+1} = f(g(B_n))$

- we can construct by bijections

$A_n \setminus A_{n+1} \to B_n \setminus B_{n+1}$

So $\bigcup_{n=0}^{\infty} A_n \setminus A_{n+1} \to \bigcup_{n=0}^{\infty} B_n \setminus B_{n+1}$

$A = \bigcup_{n=0}^{\infty} (A_n \setminus A_{n+1}) \cup \bigcap_{n=0}^{\infty} A_n$

But f is bijection

$\bigcap_{n=0}^{\infty} A_n \to \bigcap_{n=0}^{\infty} f(A_n)$

$\bigcap_{n=1}^{\infty} B_n \subset \bigcap_{n=0}^{\infty} f(A_n) \subset \bigcap_{n=0}^{\infty} B_n$

because $B_{n+1} \subset f(A_n) \subset B_n$

but $\bigcap_{1}^{\infty} B_n = \bigcap_{0}^{\infty} B_n$

because $B_{n+1} \subset B_n$

so f defines bijection

$\bigcap_{n=0}^{\infty} A_n \to \bigcap_{n=0}^{\infty} B_n$

$\text{Card } \mathbb{N} \leq \text{card } 2^{\mathbb{N}}$

$n \mapsto \{n\}$

injection

we know, $\text{card } 2^{\mathbb{N}} \neq \text{card } \mathbb{N}$

$\text{card } \mathbb{N} < \text{card } 2^{\mathbb{N}}$

$\text{Card } \mathbb{N} = \aleph_0$

$\text{Card } \mathbb{N} = \aleph_1$

- continuum

- continuum hypothesis

Real Numbers

= infinite decimal (binary)

Claim:

$\text{Card } 2^{\mathbb{N}} = \text{card } \mathbb{R}$

$(0,1)$ can be expressed as $.01000110101\ldots$

$x \longleftrightarrow$ seq. of 0 & 1

bijection between $2^{\mathbb{N}}$ and sequence of 0 & 1

$\{a_k\}_{k=1}^{\infty} \longleftrightarrow \{k : a_k = 1\}$

$\rightarrow$ this is not unique