Real Numbers

9/13/07

"Naive" definition

real # – infinite binary fraction (decimal)

Countably many exceptions (Rational #) : $\exists$ more than one representation.

ex) $0.1000\cdots$
$0.0111\cdots$

$\text{Card}([0,1] \cup \text{exceptions}) = \text{card}\left\{\{x_n\}_0^\infty : x_n = 0,1\right\} = \text{Card } 2^{\mathbb{N}}$

Collection of all binary sequences

$2^{\mathbb{N}} \overset{\varphi}{\longleftrightarrow} \{x_n\}_1^\infty$

$\varphi(\{x_n\}) = \{k \in \mathbb{N} : x_k = 1\}$

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prop X – infinite, C countable

$\Rightarrow \text{Card}(X \cup C) = \text{Card}(X)$

Proof as an exercize

← Implies that $\text{Card } \mathbb{R} = \text{Card } 2^{\mathbb{N}}$

$\text{card } 2^{\mathbb{N}} \leqslant \text{card } 2^{\mathbb{N}} \times 2^{\mathbb{N}} \cdots ①$

← Explanation that $\text{Card } \mathbb{R} \times \mathbb{R} = \text{Card } \mathbb{R}$

$A \longmapsto A \times A_0$ ← fixed (injection)

$A \subset \mathbb{N}$, $A_0 \subset \mathbb{N}$

Also, claim $2^{\mathbb{N}} \times 2^{\mathbb{N}} \subset 2^{\mathbb{N} \times \mathbb{N}}$

$A \times B \subset \mathbb{N} \times \mathbb{N}$

$A \subset \mathbb{N}$, $B \subset \mathbb{N}$

$\Rightarrow \text{Card } 2^{\mathbb{N}} \times 2^{\mathbb{N}} \leq \text{card } 2^{\mathbb{N} \times \mathbb{N}} \cdots ②$ $= \text{card } 2^{\mathbb{N}}$ because $\mathbb{N} \times \mathbb{N}$ countable

By ① & ②, $\text{card } 2^{\mathbb{N}} \times 2^{\mathbb{N}} = \text{card } \cancel{2^{\mathbb{N} \times \mathbb{N}}}\, 2^{\mathbb{N}}$

Formal def of $\mathbb{R}$

$\mathbb{R}$ — complete ordered field

field: operations $+, -, \times, \div$ (satisfying properties see textbook)

$<$ total order (only three possibilitie: $\alpha > \beta$, $\beta > \alpha$ or $\alpha = \beta$)

& agree with algebraic operations.

if $a \geq b$, $c \geq d$ $\Rightarrow$ $a + c \geq b + d$

$a > b$, $c > 0$ $\Rightarrow$ $ac > bc$

- Finite field cannot be ordered
- $\mathbb{C}$ cannot be ordered.

Complete

Let set $A \subset \mathbb{F}$ (ordered field).

A is bounded above, if $\exists M$ s.t. $\forall x \in A$ $x \leq M$

Def) $\sup A$.

The smallest upper bound of A (if exists) is called supremum of A.

$\sup A = M$ s.t. M is upperbound for A, but $\forall m < M$, $m \in A$ is not upper bound.

⊛ Complete means any non-empty bdd above A has supremum.

ex) $\mathbb{Q}$. $A = \{x \in \mathbb{Q} : x^2 < 2\}$ A does not have a rational supremum.

$\sup A = \sqrt{2}$ $\Rightarrow$ $\mathbb{Q}$ is not complete.

Why $\mathbb{R}$ is complete. (If $\mathbb{R}$ is set of binary fractions)

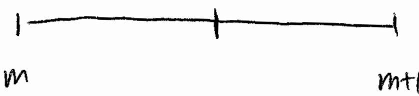
$\mathbb{R}$ - binary fractions (infinite)

$A \subset \mathbb{R}$, $A \cap (0, \infty) \neq \emptyset$ ← Assume for simplicity

Let $m = \min\{n : \forall x \in A, x < n+1\}$ m gives us integer par of sup.

$m.b_1b_2b_3\cdots$

We can find inductively all binary digits b_k of $\sup A$:



∞ ← can put ∞ here

If $A \cap [m+\frac{1}{2}, m+1) = \emptyset$ then $b_1 = 0$

$\neq \emptyset$ then $b_1 = 1$

$A \cap [m+\frac{1}{2}b_1 + \frac{1}{4}, \infty) = \emptyset$ then $b_2 = 0$

$\neq \emptyset$ then $b_2 = 1$

$\vdots$

$A \cap [m+\frac{1}{2}b_1+\frac{1}{4}, \infty)$

Next step: Check

$A \cap [m+\frac{1}{2}b_1+\frac{1}{4}b_2+\frac{1}{8},$

Thm There exists "a unique" complete ordered field.

If $\exists \mathbb{F}_1, \mathbb{F}_2$ both complete ordered field. ← "Unique" means

$\Rightarrow \exists \varphi: \mathbb{F}_1 \to \mathbb{F}_2$ bijection preserving algebraic ops & order.

Dedekind Cuts (Easiest formal construction of $\mathbb{R}$)

Cut: subset $I \subset \mathbb{Q}$ s.t.
1. I is bounded above
2. If $x \in I$, $y < x \Rightarrow y \in I$
3. I has no max element.
4. $I \neq \emptyset$, $I \neq \mathbb{Q}$

$\mathbb{Q} \cap (-\infty, x)$ ← Informally, Ded. cut is

If we know cut, we know all $q \in \mathbb{Q}$, $q < x$. So we know all $r \in \mathbb{Q}$, $r > x$

Therefore, we can approximate x by rationals with any given precision

Completeness of $\mathbb{R}$ in this def is easy.
If we have a collection A of cuts

then $$\bigcup_{\tilde{x} \in A} \tilde{x} \quad (\subset \mathbb{Q})$$

is the cut corresponding to sup
(Each $\tilde{x}$ is a subset of $\mathbb{Q}$, so is the union)
See s. 1.9 for details