MA113 9/17/2007 JUNGDO YU

Review of Sequences.

Formal def.

Sequence $a: \mathbb{N} \to \mathbb{R}$

~~traditional notation~~ :

$a(n)$ $\quad a_n \quad \{a_n\}_{n=1}^{\infty}$

Def of limit.

$a = \lim_{n\to\infty} a_n$ if

~~definition of epsilon.~~

$\forall \varepsilon > 0 \;\; \exists N \text{ s.t. } \;\; \forall n > N \;\; |a - a_n| < \varepsilon$

Limit is unique (if exists) Proof in the textbook.

Algebraic operations with limits

Thm. Let $\{a_n\}$ be bounded above ($a_n \le M \;\; \forall n$)

increasing sequence.

$\hookrightarrow \forall n \;\; a_{n+1} \ge a_n$

Then $\lim a_n = \sup\{a_n : n \in \mathbb{N}\}$

Pf. $a := \sup\{a_n : n \in \mathbb{N}\}$

Take arbitrary $(\forall)$ $\varepsilon > 0$

$a - \varepsilon$ - not an upper bound for $\{a_n\}^{\infty}$

a - upper bound

$\exists N$ s.t. $a_N > a - \varepsilon$

$\forall n > N$ $\;\; a_n \ge a_N > a - \varepsilon$

$a_n \le a < a + \varepsilon$

$a - \varepsilon < a_n < a + \varepsilon \iff |a - a_n| < \varepsilon$

Rem

$\Rightarrow \lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$

Inf(A) - greatest lower bound

Thm (Pinching theorem)

(2 cop theorem)

If $a_n \le b_n \le c_n \quad \forall n$

and $\lim a_n = \lim c_n = a$

$\Rightarrow \lim b_n = a$

Pf Take ① arb $(\forall)\ \varepsilon > 0$

$\lim a_n = a$ so,

$\exists N, \text{ s.t. } \forall n > n, \ |a - a_n| < \varepsilon$

$a - \varepsilon < a_n < a + \varepsilon$

$\lim c_n = a \Rightarrow$

$\exists N_2 \quad \forall n > N_2 \quad |a - c_n| < \varepsilon$

$a - \varepsilon < c_n < a + \varepsilon$

Put $N = \max(N_1, N_2)$

② Then $\forall n > N$ $\quad a - \varepsilon < a_n \le b_n \le c_n < a + \varepsilon$

③ $|a - b_n| < \varepsilon$

proof ends.

" Limits have nasty habit of not having

Def. $\{a_n\}_{n=1}^{\infty}$

$\overline{a_n} = \sup\{a_k : k \ge n\}$

$\underline{a_n} = \inf\{a_k : k \ge n\}$

$\overline{a_n} \searrow \quad (a_n \ge a_{n+1})$

decreasing sequence

$\overline{a_n} \nearrow \quad (a_n \le a_{n+1})$

increasing sequence

$\exists \lim \overline{a_n} = \limsup\limits_{n\to\infty} a_n = \overline{\lim}\, a_n$

$\lim \underline{a_n} = \liminf\limits_{n\to\infty} a_n = \underline{\lim}\, a_n$

For bounded sequences

limsup & liminf exist, always.

Thm If $\limsup a_n = \liminf a_n = a$

then $\lim_{n\to\infty} a_n = a$.

---

converse is also true.

Rem If $\lim a_n = a$, then ← Proof can be found in the textbook

$\limsup a_n = \liminf a_n = a$

Pf for Thm

$\underline{a_n} \le a_n \le \overline{a_n}$

let's take limits. $n \to \infty$

$\underline{a_n} \to a$, $\overline{a_n} \to a$

$\lim \overline{a_n} = \lim \underline{a_n} = a$

$\Rightarrow$ by pinching principle $\lim a_n = a$

$a_n \le b_n \le c_n$ for $\forall n$,

and $\lim a_n = \lim c_n = a$

$\Longrightarrow \lim b_n = a$

Thm (Cauchy Criterion)

$\lim a_n$ exists iff

$$\boxed{\underbrace{\forall \varepsilon > 0}_{①}\ \underbrace{\exists N}_{②} \text{ s.t. } \underbrace{\forall m,n > N}_{③}\ \underbrace{|a_n - a_m|}_{④} \underbrace{<}_{⑤} \underbrace{\varepsilon}_{⑥}}$$ — Cauchy sequence.

Pf Let $\lim a_n$ exists. Take arb ($\forall$) $\varepsilon > 0$ ①

② $\exists N$ s.t. $\forall n > N$

$|a_n - a| < \frac{\varepsilon}{2}$ → Trick: If it's true for all ε, it should be also true for $\varepsilon/2$

③ $\forall n, m > N$

④ $|a_n - a_m| = |a_n - a + a - a_m| \le |a_n - a| + |a_m - a|$

⑤ $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ ⑥ $= \varepsilon$

#4

Let $\{a_n\}$ be Cauchy

Take arb ($\forall$) $\varepsilon > 0$

$\exists N$ s.t. $\forall m, n > N$

$|a_n - a_m| < \varepsilon$

Fix $n > N$

$a_n - \varepsilon < a_m < a_n + \varepsilon \quad \forall m > N$
$\forall m \geq n$

If a_n is fixed

Take $\sup_{m \geq n} a_m$

By def of limsup.

$a_n - \varepsilon \leq \underline{a_n} \leq \overline{a_n} \leq a_n + \varepsilon$

what can be concluded from this?

$0 \leq \lim(\overline{a_n} - \underline{a_n}) \leq 2\varepsilon$

If $m \uparrow \rightarrow \overline{a_n} \downarrow$ & $\underline{a_n} \uparrow$

$\therefore$ difference gets smaller.
$\overline{a_n} - \underline{a_n}$

$0 \leq \limsup a_n - \liminf a_n \leq 2\varepsilon \quad \forall \varepsilon > 0$

when this happens?

when $\limsup a_n - \liminf a_n = 0 \Rightarrow$ * Implies there exists limit.

only zero is smaller than 'all positive epsilons'.

* This finds the existence of limit w/o finding actual limit. by using Cauchy seq.

- End -