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MA 113 Sept. 19th.

HW.

$a_n = (-1)^n + \frac{1}{n}$ lim sup $= 1$, lim inf $= -1$

Metric Spaces (6.3)

$|a - a_n| < \varepsilon \Rightarrow$ distance$(a, a_n) < \varepsilon$ ρ – metric distance on X

Def: a metric on X is a function $\rho: X \times X \to \mathbb{R}$ s.t.

1. $\rho(x,y) \geq 0 \quad \forall x,y$
2. $\rho(x,y) = 0$ iff $x = y$
3. $\rho(x,y) = \rho(y,x)$ (symmetric)
4. $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

i.e. ρ is a non-negative, non degenerate symm. fn. satisfying triangle ineq.

Ex: $\mathbb{R}^n$ $\vec{x} = (x_1, x_2, \dots x_n)$

$$\rho(x,y) = \left(\sum_{k=1}^{n} (x_k - y_k)^2\right)^{\frac{1}{2}}$$

(X, ρ) is called Metric Space

Ex: X, $f: X \to \mathbb{R}$,

$$\rho(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

Ex: $C([0,1])$

$$\rho(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)| = \max_{x \in [0,1]} |f(x) - g(x)|$$

Ex: Words (as in English words)

dist $(w_1, w_2) = \#$ of letters to change

dist (King, Kong) $= 1$

Moscow is closer by plane to Kazakhstan than the next nearest city is by train. 3hr flight < 5hr train ride

$\{X_n\}$ $X_n \in X$ is called Cauchy if $\forall \varepsilon > 0, \exists N$
$\forall n, m > N$ $\quad \rho(X_n, X_m) < \varepsilon$

Def: (X, ρ) is called complete if every Cauchy seq. has a limit.

Complete metric Spaces:

$\mathbb{R}$, $\rho(x,y) = |x-y|$

$\mathbb{R}^n \longrightarrow$ Thm: $\mathbb{R}^d$ is a complete metric space

proof: Let $\{\vec{X}(n) = (x_1(n), x_2(n), \ldots x_d(n))\}_{n=1}^{\infty}$ be Cauchy

$\forall k$ $\quad 1 \leq k \leq d$, $\quad |x_k(n) - x_k(m)| \leq \rho(\vec{x}(n), \vec{x}(m))$

$\Rightarrow \{x_k(n)\}_{n=1}^{\infty}$ is Cauchy $\therefore \exists\, a_k$ s.t $a_k = \lim_{n \to \infty} x_k(n)$

$\vec{a} = (a_1, a_2, \ldots a_d) \rightarrow$ is this the limit? Yes, proof below

Claim: $\vec{a} = \lim \vec{x}(n)$

take arb $(\forall)$ $\varepsilon > 0$

$\exists N$ s.t $\forall m, n > N \rightarrow \rho(X(n), X(m)) < \frac{\varepsilon}{2}$ ~~$\rho(x(n), a)$~~ $<$

$\left(\sum_{k=1}^{d} |x_k(n) - x_k(m)|^2 \right)^{\frac{1}{2}} < \varepsilon/2$ take lim as $m \to \infty$

you get $\left(\sum_{k=1}^{d} |x_k(n) - a_k|^2 \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{2} < \varepsilon$

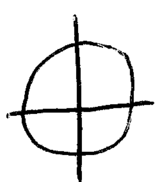
$d(\vec{x}(n), \vec{a}) \leq \frac{\varepsilon}{2} < \varepsilon$ thus $\vec{a}$ is the limit

Def: (X, ρ)

$U \subset X$ is called open if $\forall x \in U$, $\exists\, r = r(x) > 0$, s.t.

$$B_{(x,r)} \subset U$$
$$\shortparallel$$
$$\{y \in X : \rho(x,y) < r\} \subset U$$

Def: set $K \subset X$ is called closed if $K^c = \{x \in X : x \notin K\}$ is open

Ex:  $\vec{x} \in \mathbb{R}^2$ $\rho(0,x) < 1$ **Open** (disc w/o the boundary)

but if $\rho(0,x) \leq 1$ **Closed** (disc w/ boundary)

Warning: $\exists$ sets which are neither Open or Closed

$[0,1) \subset \mathbb{R}$ or partially bounded circle

$\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$