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MATH 1130

HW for today: problem 4

CLAIM:

$B_{a,r} = \{x : \rho(a,x) < r\}$

is open

PROOF:

Take arb. $(\forall)$ $x \in B_{a,r}$

Let $\varepsilon = (r - \rho(a,x))/2$

Claim: $B(x,\varepsilon) \subset B(a,r)$

Take arb. $(\forall)$ $y \in B_{x,\varepsilon}$

$\Rightarrow \rho(y,x) < \varepsilon$

$$\rho(a,y) \le \rho(a,x) + \rho(x,y)$$
$$< \rho(a,x) + \varepsilon$$
$$< r$$

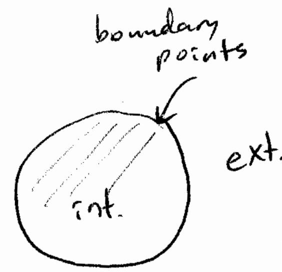
$\Rightarrow \rho(a,y) < r$

$\Rightarrow y \in B_{a,r}$

$\Rightarrow B_{x,\varepsilon} \subset B_{a,r}$

Interior, exterior and boundary

$X, \rho \qquad A \subset X$



Def $x \in A$ is called *interior* point of A if $\exists\, \varepsilon > 0$ s.t. $B_{x,\varepsilon} \subset A$

x is called *exterior* for A if x is interior for A^c

boundary point: neither exterior nor interior

U is open iff all $x \in U$ are interior

$\text{Int } A = \{x : x \text{ interior point of } A\}$

$\text{Ext } A = \{x : x \text{ is exterior for } A\}$

$\partial A = Fr = \{x : x \text{ is boundary pt. for } A\}$

<u>Prop</u>

$x \in \partial A$ iff $\forall \varepsilon > 0$ $B_{x,\varepsilon} \cap A \neq \emptyset$ and $B_{x,\varepsilon} \cap A^c \neq \emptyset$

Simply Restatement of the definition

<u>Def</u> Closure of A

$Cl A = (\text{Ext } A)^c$

$\partial A = Cl A \setminus \text{Int } A$

or $Cl A = \partial A \cup \text{Int } A$

<u>Ex</u>

$\mathbb{D} = \{x \in \mathbb{R}^2 : \rho(x,0) < 1)\}$

Int $\mathbb{D} = \mathbb{D}$

Ext $\mathbb{D} = \{x \in \mathbb{R}^2 : \rho(x,0) > 1\}$

Cl $\mathbb{D} = \{x \in \mathbb{R}^2 : \rho(x,0) \leq 1\}$

$\partial \mathbb{D} = \{x \in \mathbb{R}^2 : \rho(x,0) = 1\}$

This describes the unit disc

<u>Ex</u>

<u>$\mathbb{R}$</u>

$\mathbb{Q} \subset \mathbb{R}$

Int $\mathbb{Q} = \emptyset$, Ext $\mathbb{Q} = \emptyset$, $\partial \mathbb{Q} = \mathbb{R}$, Cl $\mathbb{Q} = \mathbb{R}$

<u>Thm</u>

Int A is the largest open subset of A.

"Largest" means any open $U \subset A \Rightarrow U \subset \text{int } A$

Proof on next page!

<u>Properties of open sets</u>

1. U_α - open $\Rightarrow \bigcup_\alpha U_\alpha$ is open
2. $U_1 \dots U_n$ - open $\Rightarrow \bigcap_1^n U_k$ - open
3. X, $\emptyset$ open

<u>Pf 1</u> $x \in \bigcup_\alpha U_\alpha \Rightarrow \exists \alpha_0$ s.t. $x \in U_{\alpha_0}$

U_{α_0} - open $\Rightarrow \exists r > 0$ s.t. $B_{x,r} \subset U_{\alpha_0} \Rightarrow B_{x,r} \subset \bigcup_\alpha U_\alpha$

<u>Pf 2</u> $x \in \bigcap_1^n U_k \Rightarrow x \in U_k$ $\forall k = 1, \dots, n$

U_k - open $\forall k$ $\exists r_k$ s.t. $B_{x,r_k} \subset U_k$

$r = \min(r_1, r_2, \dots, r_n)$, then $\forall k$ $B_{x,r} \subset B_{x,r_k} \subset U_k$

$\Rightarrow B_{x,r} \subset \bigcap_1^n U_k$

Proof (of "Int A is largest open subset of A")

1. Int A open

$x \in \text{Int } A \Rightarrow \exists r(x)$ s.t. $B_{x,r(x)} \subset A$

$\text{Int } A \subset \bigcup_{x \in \text{Int } A} B_{x,r(x)} \subset \text{Int } A$

Claim: $B_{x,r(x)} \subset \text{Int } A$

$B_{x,r(x)}$ - open

$\Rightarrow$ every pt $y \in B_{x,r(x)}$ is interior for $B_{x,r(x)} \Rightarrow y$ is interior for A

$\text{Int } A = \bigcup_{x \in \text{Int } A} B_{x,r(x)}$ - open

2. Largest Let $\underset{\text{open}}{U} \subset A$

Take arb $x \in U$, x is interior point for U

$\Rightarrow$ x is interior for a bigger set A

$\Rightarrow x \in \text{Int } A$

Thm $\text{Cl } A$ = smallest closed $K \supset A$ (essentially a corollary of the above)

$\text{Ext } A = \text{Int}(A^c)$

$\text{Cl } A = (\text{Ext } A)^c$ $\Leftarrow$ Follows from

Indeed $\text{Ext } A$ = largest open U s.t. $U \subset A^c$

equivalently = largest open U s.t. $U \cap A = \emptyset$

So $(\text{Ext } A)^c$ is smallest closed set containing A.