William Parker notes MA 113 9/24

$$Cl(A \cup B) = \underbrace{Cl(A) \cup Cl(B)}_{\text{closed, contains } A \cup B.}$$

Why minimal?

Let K closed, $A \cup B \subset K$

Then $A \subset K \Rightarrow Cl(A) \subset K$ because $Cl(A)$ is minimal closed set containing A

Then $B \subset K \Rightarrow Cl(B) \subset K$ for same reason

$\Rightarrow Cl(A) \cup Cl(B) \subset K \Rightarrow Cl(A) \cup Cl(B)$ is minimal

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TOPOLOGICAL SPACES

Def Let X, Y metric spaces and $f : X \to Y$

f is continuous at $x_0 \in X$ if $\forall \varepsilon > 0$

$\exists \delta > 0$ s.t. $\forall x \in X$, $\rho_X(x, x_0) < \delta \Rightarrow \rho_Y(f(x), f(x_0)) < \varepsilon$

Possible to give definition of continuous without ε, δ

Def An open neighborhood of $x_0 \in X$ is an open set $U \ni x_0$

Def A neighborhood of x_0 is any set containing an open neighborhood of x_0

→ Equivalently, any set A s.t. $x_0 \in \mathrm{Int}(A)$.

New definition of Continuity

f is continuous at $x_0 \in X$ if $\forall$ neighborhood $V \ni f(x_0)$,

$\exists$ neighborhood $U \ni x_0$ s.t. $f(U) \subset V$

Notation: $f^{-1}(V) = \{x \in X : f(x) \in V\}$

called inverse image of V. Not necessary that f is bijectio

Restatement

f continuous if $\forall$ neighborhood $V \ni f(x_0)$,

$x_0 \in \mathrm{Int}(f^{-1}(V))$

$\mathcal{U}$ neighborhood of $x_0 \Rightarrow x_0 \in \mathrm{Int}(\mathcal{U})$

and $\mathcal{U} \subset f^{-1}(V)$ (same as $f(\mathcal{U}) \subset V$)

$\Rightarrow x_0 \in \mathrm{Int}(f^{-1}(V))$

<u>Remark</u>: "neighborhood" can be replaced by "open neighborhood"

$\varepsilon - \delta \Rightarrow$ New definition.

Take arbitrary neighborhood $V \ni f(x_0)$.

$\Rightarrow \exists\ \varepsilon > 0$ s.t. $B_{f(x_0), \varepsilon} \subset V$.

By $\varepsilon - \delta$ def., $\exists\ \delta > 0$ s.t. $f(B_{x_0, \delta}) \subset B_{f(x_0), \varepsilon} \subset V$.

(This is just a restatement of $\varepsilon - \delta$)

Take $\mathcal{U} = B_{x_0, \delta}$. $\Rightarrow f(\mathcal{U}) \subset V$.

New def. $\Rightarrow \varepsilon - \delta$.

Take arbitrary $\varepsilon > 0$. Let $V = B_{f(x_0), \varepsilon}$

By new def., $\exists$ neighborhood $\mathcal{U} \ni x_0$ s.t. $f(\mathcal{U}) \subset V$

$\mathcal{U}$ neighborhood of x_0 means $x_0 \in \mathrm{Int}(\mathcal{U})$

$\Rightarrow \exists\ \delta > 0$ s.t. $B_{x_0, \delta} \subset \mathcal{U}$.

Then $f(B_{x_0, \delta}) \subset V = B_{f(x_0), \varepsilon}$

$\Rightarrow \forall x,\ \rho(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$.

<u>Def</u> A <u>topology</u> on a set X is a collection

$\mathcal{T} \subset 2^X$ of subsets s.t.

(1) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$

(2) $\forall\ \mathcal{U}_\alpha \in \mathcal{T}$, $\bigcup_\alpha \mathcal{U}_\alpha \in \mathcal{T}$ (arbitrary union)

(3) $\forall\ \mathcal{U}_1, \dots, \mathcal{U}_n \in \mathcal{T}$, $\bigcap_{k=1}^{n} \mathcal{U}_k \in \mathcal{T}$ (finite union)

$\mathcal{T}$ – topology, i.e. collection of open sets.

Example: open sets in a metric space form a topology

Def Int A = largest open set $U \subset A$

Take all open sets $U \subset A$, and take union

$\Rightarrow$ union is open by property #2.

Def A is neighborhood of x_0 if $x_0 \in \text{Int } A$.

Def K is closed if K^c is open

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(1) $\emptyset$, X closed

(2) K_α closed $\Rightarrow \bigcap_\alpha K_\alpha$ closed

(3) $K_1, \dots, K_n$ - closed $\Rightarrow \bigcup_{k=1}^{n} K_k$ closed

Follows from De Morgan's laws:

$\left(\bigcap_\alpha K_\alpha\right)^c = \bigcup_\alpha (K_\alpha)^c$ and K_α^c open $\Rightarrow$ union is open.

$\Rightarrow \bigcap_\alpha K_\alpha$ is closed

- similarly for (3)
- $\emptyset$, X are <u>both closed and open</u>.

Def $Cl(A)$ = smallest closed set $K \supset A$

why exists: Take all closed $K \supset A$ and take ~~arbitr~~ intersection of all K's $\Rightarrow$ intersection is closed.

Def $f: X \to Y$ is continuous if f is continuous at all $x \in X$

Prop f continuous iff $\forall$ open set $V \subset Y$, $f^{-1}(V)$ is open

"inverse image of open set is open"

- Proof in textbook

Cor f continuous iff $\forall$ closed set $K \subset Y$, $f^{-1}(K)$ is closed