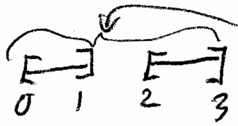
Michael Schwarz 9-28-07 (1)

Connected sets & Continuous induction


Suppose this is X.

$[0,1]$ is open in X because $[0,1] = (-1, 1.5) \cap X_0$

Def. A topological space X is called connected if the only open-closed sets are $\emptyset, X$.

Connected set $A \subset X$

A connected iff the only open-closed subsets of A (in ~~relevant~~ relative topology) are $A, \emptyset$

Interval $I \subset \mathbb{R}$ is connected (e.g. (a,b), $[a,b)$, etc.)

Prop.: If $a, b \in I \Rightarrow [a,b] \subset I$ ← Def of interval

Pf Assume $\exists\, A \subsetneq I$, A open & closed
$B = I \setminus A$ – closed & open $\Big\}$ in I

Interval is connected

assume both A and B nonempty, not all of space, so
$\exists\, a \in A,\ b \in B$
w/o loss of generality, assume $a < b$ ⇒ define seq. a_n, b_n with $a_1 = a$, $b_1 = b$

Define $c_n = \dfrac{a_n + b_n}{2}$

If $c_n \in A$ define $a_{n+1} = c_n$, $b_{n+1} = b_n$
If $c_n \in B$ define $a_{n+1} = a_n$, $b_{n+1} = c_n$

$a_n \qquad \cdot \qquad b_n$

$$\begin{array}{ll|l} a_{n+1} \geq a_n & \forall n & a_n \nearrow \\ b_{n+1} \leq b_n & \forall n & b_n \searrow \end{array}$$

$\exists \ \alpha = \lim a_n \ , \ \beta = \lim b_n$

$$\underbrace{\lim_{n\to\infty} (a_n - b_n)}_{\alpha - \beta} = \lim \frac{a-b}{2^n} = 0$$

So $\alpha = \beta$

Let $\alpha \in A$

$\exists$ open $U \subset \mathbb{R}$ s.t. $A = U \cap I$

$\Rightarrow \boxed{\exists \ \varepsilon > 0 \text{ s.t. } (\alpha - \varepsilon, \alpha + \varepsilon) \subset U} \Rightarrow \boxed{(\alpha - \varepsilon, \alpha + \varepsilon) \cap I \subset A}$

This is a "$\Rightarrow$" symbol.

$\forall n \ b_n \notin A$

$\Rightarrow b_n \notin (\alpha - \varepsilon, \alpha + \varepsilon) \cap I$

$\Rightarrow b_n \notin (\alpha - \varepsilon, \alpha + \varepsilon)$

because $b_n \in I$

$|b_n - \alpha| > \varepsilon \Rightarrow \alpha \neq \lim b_n$

Same reasoning with the show that $\alpha \notin B$

Since it was shown that $\alpha \notin A$, $\alpha \notin B$, contradiction.

__Remark__ Any connected subset of $\mathbb{R}$ is an interval.

__Prop__ A closed, $a_n \in A \Rightarrow \lim a_n \in A$

__Thm.__ $f : X \to Y$ cont., X connected $\Rightarrow f(X)$ connected

Proof

Let $A \subset f(X)$ s.t. A is open and closed

Then $f^{-1}(A)$ is open and closed

X - connected, so either:

- i) $f^{-1}(A) \neq 0 \Rightarrow A = \emptyset$ or
- ii) $f^{-1}(A) = X \Rightarrow A = f(X)$

A open in $f(X) \Rightarrow$ open in X U s.t.

$A = f(X) \cap U \quad f^{-1}(A) = f^{-1}(U)$

Remark: A is closed in X_0 if $\exists$ closed $K \subset X$

s.t. $A = X_0 \cap K$

Cor. Intermediate value thm.

(If a continuous function takes 2 values, it takes all values in between)

Def. Path in X: $\underset{\text{Continuous}}{f} : I \to X$

Def. X is called path-connected if $\forall x_1, x_2 \in X$ | $\exists$ path connecting x_1 and x_2

$\exists f : [a, b] \to X$ s.t. $f(a) = x_1$, $f(b) = x_2$

Thm X is path-connected $\Rightarrow$ X is connected

Pf. Let $A \subset X$ open & closed

$B = X \setminus A$ open & closed

Take $a \in A$, $b \in B$

Let $\underset{\text{cont}}{f} : [\alpha, \beta] \to X$ path connecting a and b.

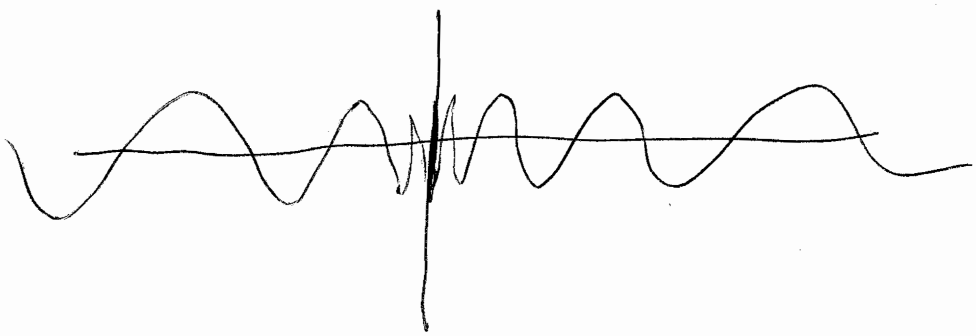
$\alpha \in f^{-1}(A)$ = open
$\beta \in f^{-1}(B)$ = open
$\Rightarrow\Leftarrow$

Here we used the fact that if $f: X \to Y$ continuous, then f treated as a function $: X \to f(X)$ is also continuous (Why?)
$f(X) \subset X$ here is endowed with relative topology

Converse is not true.

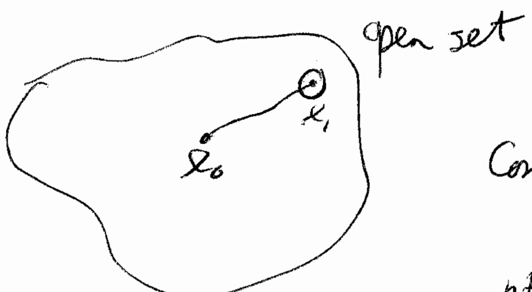
Example

graph of $y = \sin \frac{1}{x} \cup \{-1 \le y \le 1, x = 0\}$



This set is connected but not path-connected.

Thm Any open & connected subset $A \subset \mathbb{R}^d$ is path connected



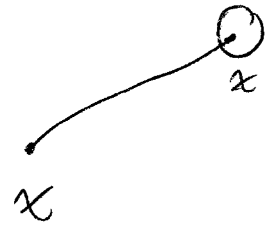
Informal idea:

Consider all pts. that can be reached by some path. All pts. have to be reached

If you can go to x_1, can get to entire ball.

Pf: Fix $x_0 \in A$ & let $A_1 = \{x \in A ; \exists \text{ path from } x_0 \text{ to } x\}$

A_1 open



because if $x \in A_1$, $\exists$ path to x,

$\exists \varepsilon > 0 \quad B_{x,\varepsilon} \subset A$

$\Rightarrow \forall y \in B_{x,\varepsilon} \; \exists$ path $x \to y$

$\Rightarrow B_{x,\varepsilon} \subset A$

Show $A \setminus A_1$ - open:

Let $x \in A \setminus A_1$

$\exists \varepsilon > 0$ s.t. $B_{x,\varepsilon} \subset A$

If $\exists \; y \in B_{x,\varepsilon}$ s.t. $y \in A_1$

$\exists$ path $x_0 \to y$

$\exists$ path $y \to x \Rightarrow \exists$ path $x_0 \to x$

$\Rightarrow \Leftarrow$

$B_{x,\varepsilon} \subset A \setminus A_1$

$\Rightarrow A \setminus A_1$ open