10/1 Gregory Thompson

Different Definitions of Connected set

A is not connected $\exists$ open $U, V \in X$ st $U \cap V = \emptyset$ (different from $U \cap V \cap A = \emptyset$)
and $A \subset U \cup V$, $U \cap A \neq \emptyset$, $V \cap A$

<u>Limits, continuity and sequences</u>

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist (DNE)

$a = \lim_{n\to\infty} x_n$ if $\forall$ neighborhood V of a $\exists N$ st $\forall n > N$ $x_n \in V$

<u>Ex</u> Connected"

$X = \{0, 1\}$ open sets $\emptyset, X, \{1\}$

Consider $x_n = 1$ for $\forall n$

claim $\lim x_n = 1$ and $\lim x_n = 0$ only 1 neighborhood of $\{0\}$ is the whole set

<u>Def</u> Topological space X is called Hausdorff
if $\forall x, y \in X$, $x \neq y$ $\exists$ open sets $U \ni x$, $V \ni y$ st $V \cap U = \emptyset$

<u>Remark</u>: a metric space is Hausdorff
$U = B_{(x, \frac{d}{2})}$ $V = B_{(y, \frac{d}{2})}$ $d = \rho(x, y)$

<u>Thm</u> if X is Hausdorff and $a = \lim x_n$, $b = \lim x_n$
then $a = b$

<u>Pf</u> Let $a \in U$, $b \in V$ U and V open $U \cap V = \emptyset$

$a = \lim_n x$ then $\exists N_1$ $\forall n > N_1$ $x_n \in U$

$b = \lim x_n$ then $\exists N_2$ st $\forall n > N_2$

$\forall n > \max(N_1, N_2) = N$

$x_n \in U$, $x_n \in V$

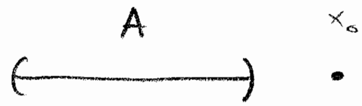
so $U \cap V \neq \emptyset$

<u>Remark</u> In Hausdorff X a singleton $(\{x\})$ is always closed

Def $X \supset A$, $x_0 \in X$

x_0 is an accumulation pt of A iff $\forall$ open $V \ni x_0$

$(V \setminus \{x_0\}) \cap A \neq \emptyset$



$f: X \supset \text{Domain } f \to Y$

$x_0 \in X$ be an accumulation point of Domain of f

$y_0 = \lim_{x \to x_0} f(x)$

If $\forall$ neighborhood V of y_0

$\exists$ neighborhood U of x_0 st $f(U \setminus \{x_0\} \cap \text{Dom}(f)) \subset V$

if

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$\forall x \in \text{Dom}(f)$ $0 < \rho(x, x_0) < \delta$

then $\rho(f(x), y_0) < \varepsilon$

Prop $f: \text{Dom}(f) \to Y$ Let x_0 be accumulatn pt of $\text{Dom}(f)$
X ? metric

Then $y_0 = \lim f(x)$ iff $\forall$ seq $\{x_n\}_1^\infty$ $\lim x_n = x_0$, $x_n \neq x_0$ $\forall n \in \mathbb{N}$

$y_0 = \lim f(x_n)$

Ex $\lim (\sin(\frac{1}{x}))$ consider $\frac{1}{x_n} = \frac{\pi}{2} + 2\pi n$

then $\lim_{n \to \infty} \sin \frac{1}{x_n} = 1$

$x_n = \frac{1}{\pi n}$

$\lim \sin \frac{1}{x_n} = 0$

Conclusion limit DNE

Pf $\lim_{x \to x_0} f(x) = y_0 \Rightarrow \lim f(x_n) = y_0$ true for X being top space

Let $y_0 \neq \lim_{x \to x_0} f(x)$ $\exists$ neighborhood V of y_0 s.t. $\forall$ neighborhood U of x_0

$\exists$ neighborhood U of x_0 st $f(U \setminus \{x_0\} \cap D) \not\subset N$ $\Leftarrow$ consider this inclusion fails

Consider $U_n = B_{x_0} \frac{1}{n}$

Because inclusion fails $\exists$ $x_n \in B_{x_0} \frac{1}{n} \cap \text{Dom}(f)$

$x_n \neq x_0$ st $f(x_n) \notin V$ $\rho(x_0, x_n) < \frac{1}{n}$

$\lim_{n \to \infty} x_n = x_0$ $f(x_n) \notin V \Rightarrow \lim_{n \to \infty} f(x_n) \neq y_0$