A. Sabelli

$$f : \underset{X}{\underset{\cap}{Dom(f)}} \longrightarrow Y$$

Remark: Thm (Limit func. / Limit Seq) is TRUE for Top. Spaces w/ "Countable base @ each point"

→ Countable base @ pt : $\forall x \in X$ $\exists$ open $V_k \ni x$
S.T. $\forall$ open $U \ni x$
$\exists$ k S.T. $V_k \subset U$
where $k \in \mathbb{N}$

Compactness

Def'n: $K \subset X$ is called "sequentially compact" if
$\forall$ seq $\{x_n\}_1^\infty$ $x_n \in K$, $\exists$ conv.
Subseq $x_{n_k} \to x_n \in K$

Def'n: K is called compact if any cover of K by open sets has a finite subcover (open cover = cover by open sets)

↳ i.e. K compact iff $\forall$ collection ②

$\{U_\alpha\}_{\alpha\in A}$ of open sets s.t. $K \subset \bigcup_\alpha U_\alpha$

$\exists\, \alpha_1 \dots \alpha_n$ s.t. $K \subset \bigcup_1^n U_{\alpha_k}$

Thm If X-metric, then $K \subset X$ is compact iff K seq. compact.

Thm (1) X-metric, $K \subset X$ is closed iff $\forall$ seq $\{x_n\}_1^\infty$ $x_n \in K$ $\lim x_n$ - exists implies $\lim x_n \in K$

(2) $\text{Clos}(A) = \{x : x = \lim_{n\to\infty} x_n,\ x_n \in A\}$

Thm $[a,b] \subset \mathbb{R}$ compact

Pf $x_n \in [a,b] \implies \{x_n\}$ - bdd

$\implies \limsup x_n$ exists

$\implies \exists\, \{x_{n_k}\} \to \limsup x_n \in [a,b]$

Thm X - compact, K - closed $\subset X$

$\longrightarrow K$ is compact

Pf let U_α - open sets

$\bigcup_\alpha U_\alpha \supset K$

$\bigcup_\alpha U_\alpha \cup K^c = X$

$\longrightarrow \exists$ finite subcover

$\bigcup_{k=1}^{n} U_{\alpha_k} \cup K_c = X$

Then $\bigcup_{k=1}^{n} U_{\alpha_k} \supset K$

Thm X is Hausdorff, K - compact $\subset X$

$\longrightarrow K$ - closed

Pf: Fix $x \notin K$, $y \in K$: $\forall y \in K$ $\exists$ open U_y

s.t. $U_y \ni y$ & $V_y \ni x$ s.t. $U_y \cap V_y = \emptyset$

$K \subset \bigcup_{y \in K} U_y$ — open cover

$\rightarrow \exists\ y_1 \dots y_n$ s.t. $K \subset \bigcup_1^n U_{y_k}$

$x \in \bigcap_1^n V_{y_k}$ - open

$\bigcap_1^n V_{y_k} \cap \left(\bigcup_1^n U_{y_k} \right) = \emptyset$

$\Longrightarrow \bigcap V_{y_k} \cap K = \emptyset$

So x - interior pt of K^c

(x was arbitrary, so K^c open)

<u>Thm</u>| $f: X \to Y$ K-compact $\subset X$ f-cont.

$\rightarrow f(K)$ - compact

note: $\underbrace{f(\text{closed})}_{\text{in } \mathbb{R} \text{ must be unbounded}} \rightarrow$ not closed

Def'n $K \subset X$ metric is bdd if
$\exists\, x_0, R$, $K \subset B_{x_0,R}$

thm K compact $\subset X$ metric $\Rightarrow K$ bdd

pf fix x_0, $\bigcup_1^\infty B_{x_0,n} = X \supset K$

$\Rightarrow \exists\, n_1 < n_2 < \dots < n_m$ s.t. $K \subset \bigcup_1^m B_{x_0,n_k}$

$= B_{x_0,n_m}$

Thm $K \subset \mathbb{R}$ compact iff K closed and bounded

K closed & bdd $\to K \subset [a,b]$

<u>Thm</u> | $\underset{\text{cont}}{f} : K \to \mathbb{R}$, K compact

$\Longrightarrow \exists\ x_{max}, x_{min}$ s.t.

$\forall x \in K \quad f(x_{min}) \leq f(x) \leq f(x_{max})$

<u>Pf</u>

$f(K)$ compact

<u>Prop</u>: $\underset{\text{compact}}{R} \subset \mathbb{R} \to \sup(R) \in R$
$\inf(R) \in R$

Note Typical Mistake: General compact set is not closed & Bdd (is true in $\mathbb{R}$)