10/5/07

Product Topology & Compactness

$\mathcal{X}, Y$ - top

$\tau_{\mathcal{X}}\ \tau_Y$

$\mathcal{X} \times Y$

<u>Def</u>: Base of product topology on $\mathcal{X} \times Y$:

$\{ U \times V : U \in \tau_{\mathcal{X}} , V \in \tau_Y \}$

<u>Alternatively</u>: if $\mathcal{B}_{\mathcal{X}} \subset \tau_{\mathcal{X}}$ is a base of $\tau_{\mathcal{X}}$

and $\mathcal{B}_Y \subset \tau_Y$ is a base of τ_Y

then $\{ U \times V : U \in \mathcal{B}_{\mathcal{X}} , V \in \mathcal{B}_Y \}$ is a base of product topology.

<u>Ex</u>: $\mathcal{X} = Y = \mathbb{R}$

(a,b) - base of $\tau_{\mathbb{R}}$

base of $\mathcal{X} \times Y = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$: intervals $(a,b) \times (c,d)$ base of $\tau_{\mathbb{R}\times\mathbb{R}}$

This collection of sets is a base of standard (metric) top in $\mathbb{R}^2$

Recall <u>Thm</u>: $\mathcal{B} \subset \tau \subset 2^{\mathcal{X}}$ is a base of top τ if $\forall x \in \mathcal{X}$, $\forall U \in \tau$ $x \in U$ $\exists$ $V \in \mathcal{B}$ s.t. $x \in V \subset U$.

<u>Prop</u>: Metric top in $\mathbb{R}^2$ = product top.

<u>Thm</u>: $(x_n, y_n) \in \mathcal{X} \times Y$ then $(x_n, y_n) \to (x, y)$ in product top iff $x_n \to x$ & $y_n \to y$ in respective topologies.

Thm (Baby Tickhonov's Thm): If A and B are compact then $A \times B$ is compact. (in product topology)

Pf: (only for metric spaces)

Let $\{(x_n, y_n)\}_1^\infty$ seq. in $A \times B$.

A is compact $\Rightarrow \exists\ x_{n_k} \to x_0 \in A$.

$y_{n_k} \in B$, B is compact

$\Rightarrow \exists$ conv $\{y_{n_{k_j}}\}_1^\infty$ s.t. $y_{n_{k_j}} \xrightarrow[j\to\infty]{} y_0 \in B$

$x_{n_{k_j}} \to x_0$ because a subsequence of a convergent seq. has the same lim.

$\Rightarrow (x_{n_{k_j}}, y_{n_{k_j}}) \to (x, y) \in A \times B$ $\square$

Recall: A homeomorphism is a bijection s.t. f, f^{-1} are both continuous

X & Y are homeomorphic if $\exists$ a homeomorphism $f: X \to Y$

Ex: ① $[0,1]$ not homeomorphic to $(0,1)$ since $[0,1]$ is compact and $(0,1)$ is not.

② $(0,1)$ is homeomorphic to $\mathbb{R}$ (tangent)

③ $\mathbb{R}$ is not homeomorphic to $\mathbb{R}$:

if $f: \mathbb{R}^2 \to \mathbb{R}$ is homeomorphic

then $f: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \setminus \{f(0)\}$ is also homeomorphic,

but $\mathbb{R}^2 \setminus \{0\}$ is connected and $\mathbb{R} \setminus \{f(0)\}$ is not.

④ For the same reason the circle is not homeomorphic to the interval.

Thm: Let X, Y be Hausdorff and compact

$f: X \to Y$ is continuous bijection.

Then f is a homeomorphism.

Pf: Let $\underset{\text{closed}}{K} \subset X$ K is closed, X is compact $\Rightarrow$ K is compact

f is continuous, K is compact $\Rightarrow$ $f(K)$ is compact

Y is Hausdorff, $f(K)$ is compact

$\Rightarrow$ $f(K)$ is closed. $\forall$ closed K, $f(K)$ is closed

since f is bijection, $\forall$ open $\mathcal{U}$, $f(\mathcal{U})$ is open

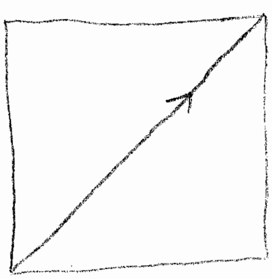
since $f(K^c) = f(K)^c$

so f^{-1} is continuous.

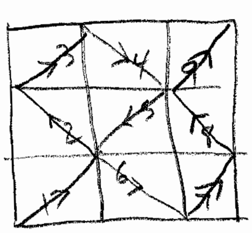
Peano Curve

Cont: $f: [0,1] \to [0,1] \times [0,1]$

s.t. $f([0,1]) = [0,1] \times [0,1]$



$f_1(t)$ $f_1 : [0,1] \to \mathbb{R}$



$f_2(t)$ We replace this more complicated path recursively in each smaller square.

$$\lim_{n \to \infty} f_n = f$$