David Horsberger

Midterm

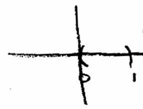
2 parts. 1^{st} part – several simple problems. closed notes.

) about 50% each.

will maybe ask for a definition. apply definitions
for example, recognize open and closed sets.

relative topology

$(0,1)$ open in $\mathbb{R}$



on $\mathbb{R}^2$ – not open or closed.

compute interior, exterior, closure.

simple proofs. (usually 1 step application of definition).

take-home part

more complicated problems.
like homework.
Due Monday.

Be able to recognize:
- Cardinality – countable, uncountable sets, sets of equal cardinality ($\text{card} X < \text{card}\, 2^X$)
- topology – open, closed, compact, connected, path connected.

compute interior, exterior closure.
what properties are preserved under direct, inverse image (continuous function)

direct: compactness, connectedness, path-connectedness

inverse: open, closed.

HW

find a set that is closed and bounded but not compact.
not in $\mathbb{R}^n$

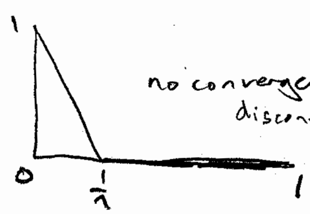
$C[0,1]$

space of continuous functions on $[0,1]$

You can do this with any ∞ dimensional space.

complete metric space $d(f,g) = \max_{x\in[0,1]} |f(x)-g(x)|$

$B_{0,1} = \{f \in C[0,1] : |f(x)| \leq 1 \; \forall x\}$ not compact.



no convergent subsequence.
discontinuous limit.

Differentiation

$f(X) = X^2$

$X \in M_{n\times n}$ (matrix)

$f'(X) \neq 2X$ (but will be $2X$ in the case of a 1×1 matrix)

$f: \Omega \to \mathbb{R}^m$

$\Omega \subset \mathbb{R}^n$ open

$x \in \Omega$

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ doesn't work for vectors

Def: f is differentiable at x if $\exists$ linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $r(h) = f(x+h) - f(x) - L(h)$ satisfies $\lim_{|h|\to 0} \frac{r(h)}{|h|}$

$|h| = \|h\| = \sqrt{h_1^2 + h_2^2 + \dots h_n^2}$

euclidean norm

O and o

$f = o(g)$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{|g(x)|} = 0$

assume $g(x) = 0 \Rightarrow f(x) = 0$

$f = O(g)$ if $\exists$ neigh $U \ni x_0$ and $C < \infty$ (constant) s.t. $f(x) \leq C g(x)$ $\forall x \in U$

f is differentiable iff $f(\vec{x+h}) = f(x) + \vec{L}(h) + o(\vec{h})$, $\vec{h} \to 0$

differentiable if in small scale it can be approximated by an affine function

Linear: $L(\alpha \vec{h}_1 + \beta \vec{h}_2) = \alpha L(\vec{h}_1) + \beta L(\vec{h}_2)$

Linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ are $m\times n$ matrices. They are the same thing. Formally, though, derivative is the matrix.

Def: L called differential of f at x. written $df(x)$

Matrix of df called derivative

Derivative of $f(x)=x^2$

$$f(x+h)$$
$$=(x+h)^2=(x+h)(x+h)$$
$$=\underbrace{x^2}_{f(x)}+\underbrace{hx+xh}_{L(h)}+\underbrace{h^2}_{o(h)}$$

linear part depends on x.

$$\lim_{h\to 0}\frac{h^2}{|h|}=\lim_{h\to 0}\overbrace{\frac{h}{|h|}}^{\text{bounded}}\cdot\underbrace{h}_{\text{goes to } 0.}=0$$

$$\underbrace{O(1)}_{\text{means bounded}}\cdot\underbrace{o(1)}_{\text{means goes to } 0.}=o(1)$$

derivative is a function that takes $\underset{\substack{\Uparrow \\ M_{n\times n}}}{h}\mapsto hX+Xh$

$$(dX)^2(h)=hX+Xh$$

ΔX instead of h.