Ma113 Notes - 10/15/07

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Linear Algebra Review

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\vec{x} \in \mathbb{R}^n$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

T linear means $T(\alpha\vec{x} + \beta\vec{y}) = \alpha T\vec{x} + \beta T\vec{y} \quad \forall \alpha, \beta \in \mathbb{R}, \ \forall \vec{x}, \vec{y}$

For T linear transformation associate matrix $\{T_{i,j}\}_{i=1}^{m}{}_{j=1}^{n}$ ie) $\begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\ T_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ T_{m,1} & \cdots & \cdots & T_{m,n} \end{pmatrix}$

Formally,

T linear Transformation

$[T]$ its matrix

However, linear Transformation and its matrix used interchangably. $\boxed{T\vec{x}}$

Remark: $T_{i,j}$ in terms of linear Transformation

Standard basis $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$ $\vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{e}_1 + \ldots + x_n\vec{e}_n$

and $T_{i,j} =$ coordinate $\#i$ of $T\vec{e}_j$

Also say $T\vec{e}_j =$ column $\#j$ of matrix of T...

Norm

$$|\vec{x}| = \|\vec{x}\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

1) $\|\vec{x}\| \geq 0$ "non-negative"
2) $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$ "non-degenerate"
3) $\|\alpha \vec{x}\| = |\alpha| \cdot \|\vec{x}\|$
4) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ "triangle inequality"

Any function $\vec{x} \to \|\vec{x}\| \in \mathbb{R}$ sat. 1–4 is called a <u>norm.</u>

Example

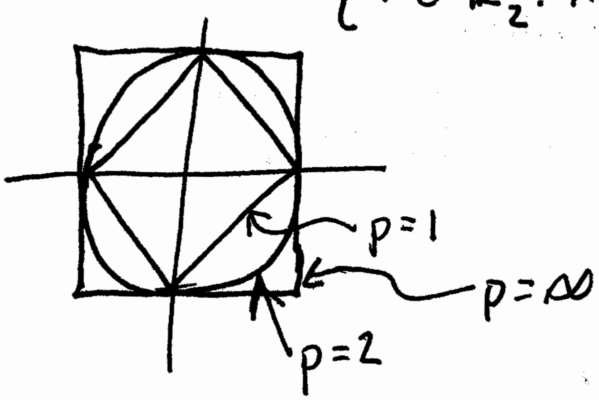
Should be $|x_k|^p$?

$$\|\vec{x}\|_p = \left(\sum_{k=1}^{n} |x|_k^p\right)^{1/p}$$

$$1 \leq p < \infty \qquad \|\vec{x}\|_\infty = \max_k |x_k|$$

Picture $\mathbb{R}^2$

$$\{\vec{x} \in \mathbb{R}_2 : \|\vec{x}\| \leq 1\}$$



Ball under different norms

$p=2$ is "euclidean norm"

All these norms $\|\ \|_p$ are equivalent.

2 Norms $\|\ \|$ and $\|\ \|_*$ are called equivalent if $\exists\, C < \infty$ st

$$\frac{1}{C}\|\vec{x}\| \leq \|\vec{x}\|_* \leq C\|\vec{x}\|$$

Note: $C=n$ for p-norms in $\mathbb{R}^n$

if we have a norm we can define $\rho(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$ which gives us a topology.

Observation: Equivalent Norms give identical Topologies. (Trivial)

If we change norm $\|\cdot\|$ to an equivalent $\|\cdot\|_*$ the differentiability & derivative does not change.

Pf $f(\vec{x}+\vec{h}) - f(\vec{x}) - L\vec{h} =: r(\vec{h})$

$$\lim_{h\to 0} \frac{r(\vec{h})}{\|h\|} = 0$$

but $h\to 0$ is equivalent to $\|h\|\to 0$ so,

now look at

$$\lim_{\|h\|\to 0} \frac{r(\vec{h})}{\|\vec{h}\|_*} \Rightarrow \lim_{\|h\|\to 0} \underbrace{\frac{r(\vec{h})}{\|\vec{h}\|}}_{\to 0} \cdot \underbrace{\frac{\|\vec{h}\|}{\|\vec{h}\|_*}}_{\le C}$$

$\|h\|$ not $\|h\|_*$

$= 0$

bounded $\cdot$ ($\lim = 0$) = ($\lim = 0$)

$$O(1) \cdot o(1) = o(1)$$

Thus, to reiterate, differentiability & derivative does not change

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Thm: Any 2 norms on $\mathbb{R}^n$ are equivalent ("Very Advanced")

Compute $df(x)$

$f: \Omega \to \mathbb{R}^m$, $\quad \Omega \subset \mathbb{R}^n$ open $\qquad x \in \Omega$

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} \qquad f_k : \Omega \to \mathbb{R}$$

$$f(\vec{x}+\vec{h}) = f(\vec{x}) + L \cdot \vec{h} + o(h) \quad \leftarrow \text{By Definition}$$

$$\boxed{\text{let } r(\vec{h}) = o(\vec{h})}$$

$$\vec{h} = h \cdot \vec{e}_j$$

$$f_i(\vec{x}+h\vec{e}_j) = f_i(\vec{x}) + (Lh\vec{e}_j)_i + r_i(h\vec{e}_j)$$

$$\frac{f_i(\vec{x}+h\vec{e}_j) - f_i(\vec{x})}{h} = \frac{(L\cancel{h}\vec{e}_j)_i}{\cancel{h}} + \frac{r_i(h\vec{e}_j)}{h}$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{\partial f_i}{\partial x_j} = (L\vec{e}_j)_i + 0$$

$$0 \le \frac{|r_i(h\vec{e}_j)|}{h} \le \frac{\|r(h\vec{e}_j)\|}{\|h e_j\|} \;\xrightarrow{h\to 0}\; 0$$

$$\boxed{\frac{d f_i}{d x_j} = L_{ij}}$$

As a result, Matrix of $\delta f(x) = \left\{ \frac{\partial f_i}{\partial x_j} \right\}_{i=1 \; j=1}^{m \;\; n}$

Called Jacobi matrix