10/19/07 # Norms of Matricies
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$M_{m \times n} \simeq \mathbb{R}^{m \cdot n}$

sum of diagonal entries

$$\|A\|_{\sigma_2} = \left(\sum |a_{ij}|^2\right)^{1/2} = \left[tr(A^T A)\right]^{1/2}$$

Frobenius norm
Hilbert-Schmidt

$$= \left[tr(A^* A)\right]^{1/2}$$

$$A^* = \overline{A^T}$$

Operator norm

$$\sup_{\|\vec{x}\| \le 1} \|A\vec{x}\| = \|A\|_{op}$$

for $\vec{x} \in \mathbb{R}$, $\|\vec{x}\| = \|\vec{x}\|_2$

$$\|A\|_{op} = \underbrace{\begin{pmatrix}\text{Maximal} \\ \text{e-value of } A^*A\end{pmatrix}^{1/2}}_{\text{singular value}}$$

$$\|A\|_{op} \le \|A\|_{\sigma_2}$$

max of e.values — sum of e. values

trace = sum of e.-values counting with multi.
e. values of A^*A are ≥ 0

$$\|A\vec{x}\| \le \|A\| \cdot \|\vec{x}\|$$

$$\|A\vec{x}\| = \left\| \|\vec{x}\| A \frac{\vec{x}}{\|\vec{x}\|} \right\| \le \|\vec{x}\| \underbrace{\left\| A \frac{\vec{x}}{\|\vec{x}\|} \right\|}_{\le \|A\|_{op}}$$

$$\|AB\| \le \|A\| \cdot \|B\|$$

True for both types of norms

Thm: (Chain rule)

f differentiable at $\vec{x}$

g diff. at $\vec{y} = f(\vec{x})$

Then $g \circ f = g(f)$ is diff. at $\vec{x}$ and

$[g \circ f]'(\vec{x}) = g'(\vec{y}) \cdot f'(\vec{x})$

Proof: $f(\vec{x}+\Delta\vec{x}) = f(\vec{x}) + \underbrace{f'(\vec{x}) \cdot \Delta\vec{x} + r_1(\Delta\vec{x})}_{\Delta\vec{y}}$, $r_1(\Delta\vec{x}) = o(\Delta\vec{x})$

$\vec{h} = \Delta\vec{x}$

$$g(\vec{y}+\Delta\vec{y}) = g(\vec{y}) + g'(\vec{y})\Delta\vec{y} + r_2(\Delta\vec{y})$$

$r_2 = o(\Delta\vec{y})$

$$g(f(\vec{x}+\Delta\vec{x})) = g(f(x)) + g'(\vec{y}) \cdot f'(\vec{x})\Delta\vec{x} \underbrace{+ g'(\vec{y}) \cdot r_1(\Delta\vec{x}) + r_2(f'(\vec{x})\Delta\vec{x} + r_1(\Delta\vec{x}))}_{o(\Delta\vec{x})}$$

$$\lim_{\|\Delta\vec{x}\| \to 0} \frac{\|g'(\vec{y}) \cdot r_1(\Delta\vec{x})\|}{\|\Delta\vec{x}\|} \leq \frac{\|g'(\vec{y})\| \cdot \|r_1(\Delta\vec{x})\|}{\|\Delta\vec{x}\|} \to 0$$

$$\frac{\|r_2(\dots)\|}{\|\Delta\vec{x}\|}$$

$$\frac{r_2(f'(\vec{x})\Delta\vec{x} + r_1(\Delta\vec{x}))}{\|f'(x)\Delta\vec{x} + r_2(\Delta\vec{x})\|} \cdot \frac{\|f'(\vec{x})\Delta\vec{x} + r_1(\Delta\vec{x})\|}{\|\Delta\vec{x}\|}$$

$$= \underbrace{\frac{\|r_2(\Delta\vec{y})\|}{\|\Delta\vec{y}\|}}_{\to 0} \cdot \frac{\|\Delta\vec{y}\|}{\|\Delta\vec{x}\|} = O(1)$$, so total lim is 0

$\|\Delta\vec{x}\| \to 0$

$\Rightarrow \|\Delta\vec{y}\| \to 0$

$\Delta\vec{y} = O(\Delta\vec{x})$

$f'(\vec{x}) \cdot \Delta\vec{x} = O(\Delta\vec{x})$ $\quad (\|f'(x)\Delta\vec{x}\| \leq \|f'(x)\| \cdot \|\Delta\vec{x}\|)$

$r_1(\Delta\vec{x}) = o(\Delta\vec{x})$
