Notes
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$$X + \Delta X = X(I + X^{-1}\Delta X)$$

$$(X + \Delta X)^{-1} = (\underbrace{I + X^{-1}\Delta X}_{A})^{-1} X^{-1}$$

Recall:

$$\boxed{(I + A)^{-1} = I - A + A^2 - A^3 + A^4 \cdots \quad \|A\| < 1}$$

f diff at $\vec{x}_0 \Rightarrow f$ cont at $\vec{x}_0$

$\not\Uparrow \quad \Downarrow \quad\quad \not\Leftarrow$

all $\frac{\partial f}{\partial x_k}$ exist at $\vec{x}_0$

See 8b, P196

all $\frac{\partial f}{\partial x_k}$ exist in a neigh of $\vec{x}_0$ and cont at $\vec{x}_0 \Rightarrow f$ diff at $\vec{x}_0$

all dir der exist at $x_0 \not\Rightarrow f$ diff at x_0

Mean Value Theorem

1D. Thm f differentiable on $(x, x+h)$, cont on $[x, x+h]$

Then $\exists\, \theta \in (0,1)$ s.t $f(x+h) - f(x) = f'(x + \theta h)\cdot h$

Thm $f : \underset{\mathbb{R}^n}{\Omega \subset} \mapsto \mathbb{R}$

$(\vec{x} \in \Omega,\ \vec{h} \in \mathbb{R}^n)$

f is diff in Ω

$\vec{x}, \vec{h}$ be such that $\vec{x} + t\vec{h} \in \Omega \ \forall\, t \in [0,1]$

Then $\underset{\exists \theta \in (0,1)}{\wedge}$ $f(\vec{x}+\vec{h}) - f(\vec{x}) = f'(\vec{x}+\theta\vec{h})\cdot\vec{h}$

Pf

$g(t) = f(\vec{x} + t\vec{h}) \quad t \in [0,1]$

$g'(t) = f'(\vec{x} + t\vec{h})\,\vec{h}$

$f' = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

$\vec{h} = (h_1, \dots, h_n)^T$

$\exists\, \theta \in (0,1)$ s.t $g(1) - g(0) = g'(\theta) \cdot 1$

Remark! Theorem as stated fails for functions with values in $\mathbb{R}^n$, $n > 1$

$f : \mathbb{R} \to \mathbb{R}^2 \quad f(t) = [\cos t, \sin t]^T \qquad f'(t) = (-\sin t, \cos t)^T$

$f(2\pi) - f(0) = 0$ but $\|f'(t)\| = 1$

Thm $f : \Omega \to \mathbb{R}^m \quad m > 1$, $\Omega \subset \mathbb{R}^n$

Assume is diff in Ω $\vec{x}, \vec{h}$ s.t $\vec{x} + t\vec{h} \in \Omega \quad \forall t \in [0,1]$

$\vec{x} + t\vec{h} \in \Omega \quad \forall \ t \in [0,1]$

Then $\|f(\vec{x}+\vec{h}) - f(\vec{x})\| \le \sup_{t \in [0,1]} \|f'(x+th)\| \cdot \|\vec{h}\|$

Pf. $f(\vec{x}+\vec{h}) - f(\vec{x}) \neq \vec{0}$

$$\vec{u} = \frac{f(\vec{x}+\vec{h}) - f(\vec{x})}{\|f(\vec{x}+\vec{h}) - f(\vec{x})\|} \qquad \|\vec{u}\| = 1$$

$$\langle \vec{x}, \vec{y} \rangle = \sum_{k=1}^{n} x_k y_k$$

$$g(\vec{y}) = \langle f(\vec{y}), \vec{u} \rangle$$

$$g : \Omega \to \mathbb{R}$$ (apply previous shown MVT for scalar valued funcs)

$$\exists \ \theta \in (0,1) \ \text{s.t} \ \underbrace{g(\vec{x}+\vec{h}) - g(\vec{x})}_{\langle f(\vec{x}+\vec{h}) - f(\vec{x}), \vec{u} \rangle} = g'(\vec{x}+\theta\vec{h}) \cdot \vec{h}$$

$f(t) = \langle f(\vec{x}+t\vec{h}), \vec{u} \rangle$

$f'(t) = f'(\vec{x}+t\vec{h})\vec{h}, \vec{u}$

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \langle f'(\vec{x}+\theta\vec{h})\vec{h}, \vec{u} \rangle$$

$$f(1) - f(0) = f'(\theta) \quad (\exists \theta)$$

$$\| f'(\theta) \| \le \| f'(x+\theta h) \| \cdot \| \vec{h} \| \cdot \underbrace{\| \vec{u} \|}_{1}$$