Higher order derivatives & Taylor's formula

Def $f \in C^1$ in Ω

if $f'(x)$ exists $\forall \vec{x} \in \Omega$

and $x \to f'(x)$ is continuous in Ω

$\Updownarrow$

all $\frac{\partial f_k}{\partial x_j}$ are continuous in Ω

$f: \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \longrightarrow \mathbb{R}^m$

$f'(\vec{x}) \in L(\mathbb{R}^n \to \mathbb{R}^m)$

$f': \Omega \to L(\mathbb{R}^n \to \mathbb{R}^m)$

$\vec{x} \mapsto f'(\vec{x})$

Take derivative of f'

$d^2 f \quad f'' \in \underbrace{L(\mathbb{R}^n \to L(\mathbb{R}^n \to \mathbb{R}^m))}$

$\cong$

All bilinear forms
$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$

$B(h_1, h_2)$. Linear in each argument

$m=1 \quad f: \underset{\mathbb{R}^n}{\overset{\cap}{\Omega}} \longrightarrow \mathbb{R}$

$f'(\vec{x}) = \left(\frac{\partial f}{\partial x_1} \; \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}\right), \quad f'' = \left\{\frac{\partial^2 f}{\partial x_k \partial x_j}\right\}_{k,j=1}^{n}$ ← Hessian

$f''' = \left\{\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\right\}_{i,j,k=1}^{n}$

$f'' \quad \vec{h}^1 = (h_1^1 \dots h_n^1) \quad \vec{h}^2 = (h_1^2 \dots h_n^2) \quad \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} h_j^1 h_k^2$ ← Bilinear form

---

__Def__: $f \in C^r(\Omega)$

If all partials of order r exist & cont in Ω

__Thm__: $f \in C^2(\Omega)$ then: $\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j}$

Taylor formula (one variable)

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + \frac{f^{(n+1)}(x+\theta h)}{(n+1)!} h^{n+1}$$

$\theta \in (0,1)$

if $f \in C^{n+1}$

$f \in C^n$

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + o(h^n)$$

$m=1 \quad f: \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \longrightarrow \mathbb{R}$

$$g(t) = f(\vec{x} + t\vec{h}) \qquad t \in \mathbb{R}$$

$$g'(t) = \sum_{i=1}^{n} D_i(\vec{x} + t\vec{h})\, h_i \qquad D_i f = \frac{\partial f}{\partial x_i}$$

$$= f' \quad \vec{h}$$

$$(D_1 f \; D_2 f \ldots D_n f) \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$g''(t) = \sum_{i=1, j=1}^{n\,n} D_j D_i f(\vec{x} + t\vec{h})\, h_i h_j$$

$$g^{(k)}(t) = \sum_{i_1, i_2 \ldots i_k = 1}^{n} D_{i_1} D_{i_2} \ldots D_{i_k} f(x+th)\, h_{i_1} h_{i_2} \ldots h_{i_k}$$

n^k - terms.

$$d^k f_{\vec{x} + t\vec{h}}[h]$$

Thm $f: \underset{\mathbb{R}^n}{\Omega} \to \mathbb{R}$, $f \in C^{N+1}(\Omega)$

$\vec{x}_0, \vec{h} \in \mathbb{R}^n$ are such that $\vec{x}_0 + t\vec{h} \in \Omega$ $\forall t \in [0,1]$.

Then $$f(\vec{x}+\vec{h}) = \sum_{k=0}^{N} \frac{d^k f_{\vec{x}}[h]}{k!} + \frac{d^{N+1} f_{\vec{x}+\theta\vec{h}}[h]}{(N+1)!}$$

$d^0 f = f$, $0! = 1$.

Thm: $f \in C^N(\Omega)$, $\vec{x} \in \Omega$

~~Then~~ Then $$f(x+h) = \sum_{k=0}^{N} \frac{d^k f_{\vec{x}}[\vec{h}]}{k!} + \underbrace{o(\|h\|^N)}_{= o(\|h\|^N)}$$

for suff small h.

Proof: exercise, look at $\|h\| \to 0$

Multindex notation

$\alpha = (\alpha_1 \alpha_2 \ldots \alpha_n)$ $\alpha_k \in \mathbb{Z}_+$

$\vec{x} = (x_1, x_2 \ldots x_n) \in \mathbb{R}^n$

$\vec{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

$D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} f$

$\frac{\partial^{|\alpha|}}{\partial x^\alpha} f$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$

$\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!$ $(0! = 1)$

$|\alpha| = k$
$\alpha = (\alpha_1 \ldots \alpha_n)$
$D^\alpha f(\vec{x}) h^\alpha$ $\quad \frac{k!}{\alpha!}$

$k=2$ $\sum D_{i_1} D_{i_2} f h_{i_1} h_{i_2}$

1 if $i_1 = i_2$

2 if $i_1 \neq i_2$