Sufficient condition of Extremum 10/26/07

$f : \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \to \mathbb{R}$ $\quad f \in C^2(\Omega)$

$\vec{x}_0 \in \Omega \quad f'(\vec{x}_0) = \vec{0}$

If $d^2 f_{\vec{x}_0} > 0$ $\left(d f^2_{\vec{x}_0}[\vec{h}] > 0 \ \forall \vec{h} \neq \vec{0} \right)$, then f $\underbrace{\text{has local}}_{\text{(strict)}}$ minimum at $\vec{x}_0$

Idea of the proof,

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \frac{d^2 f_{\vec{x}_0}[\vec{h}]}{2} + r(\vec{h}) \quad \text{Where} \quad r(\vec{h}) = o(\|\vec{h}\|^2)$$

If $d^2 f_{\vec{x}_0} > 0$, then $\underset{\exists c > 0}{\widehat{\ }}$ $d^2 f_{\vec{x}_0}[\vec{h}] \geq c \|\vec{h}\|^2$

$C =$ Smallest eigen value of $\left\{ \frac{\partial^2 f}{\partial x_k \partial x_j} \right\}_{k,j=1}^{n}$ at $\vec{x} = \vec{x}_0$

$c = \inf\limits_{\|h\|=1} d^2 f_{\vec{x}_0}[\vec{h}] = \min\limits_{\|h\|=1} d^2 f_{\vec{x}_0}[\vec{h}] > 0 \longleftarrow \left(d f^2_{\vec{x}_0}[\vec{h}] > 0 \ \forall \vec{h} \neq 0 \right)$

Cont. and compact quadratic function

Using the definition of o

Find $\delta > 0$ s.t. $\forall \vec{h} \in \mathbb{R}^n \quad \|\vec{h}\| < \delta \Rightarrow |r(\vec{h})| \leq \frac{c}{2} \|\vec{h}\| \leq d^2 f_{\vec{x}_0}[\vec{h}]$

And rest of the proof done by Triangular inequality.

Inverse Function and Implicit Function Thm

$f : \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \to \mathbb{R}^n \qquad f(\vec{x}) = \vec{y} \qquad \vec{y} \in \mathbb{R}^n$

Thm Inverse Function Thm

$f : \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \to \mathbb{R}^n$, $\quad f \in C^r(\Omega)$

$\vec{x}_0 \in \Omega$ st. $f'(\vec{x}_0)$ not singular (non-zero determinant), let $\vec{y}_0 = f(\vec{x}_0)$

Then $\exists$ open neighborhood U of $\vec{x}_0$ st. $f(U)$ open f is a biject $U \to f(U)$, and $g = f^{-1} : f(U) \to \mathbb{R}^n$ $g \in C^r$ $\lrcorner$

Compute $g' = (f^{-1})'$

$g(f(\vec{x})) = \vec{x}$ $\quad g'(f(\vec{x})) \cdot f'(x) = I$ $\quad\quad \varphi(\vec{x}) = \vec{x}$ $\quad \varphi(\vec{x} + \vec{h}) = \vec{x} + \vec{h}$

$(\ d\varphi_{\vec{x}} = Id\ ,\ Id(\vec{h}) = \vec{h}\ )$

$g'(\vec{y}) = (f'(\vec{x}))^{-1}$ where $\vec{y} = f(\vec{x})$

Implicit Function

$F(x,y) = \vec{0} \in \mathbb{R}^n$ $\quad x, y \in \mathbb{R}^n$

$$F_{\vec{x}} = \left\{ \frac{\partial F_j}{\partial x_i} \right\}_{i=1\ j=1}^{n\ \ m} \qquad F_{\vec{y}} = \left\{ \frac{\partial F_j}{\partial y_i} \right\}_{i=1\ j=1}^{n\ \ m}$$

[Implicit Function Thm]

$F : \Omega \to \mathbb{R}^n$, $F \in C^r$

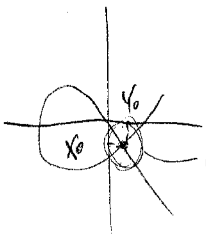
$\Omega \subset \underset{\vec{x}}{\mathbb{R}^n} \times \underset{\vec{y}}{\mathbb{R}^m}$ $\quad$ Assume $F(\vec{x}_0, \vec{y}_0) = \vec{0}$, $F_{\vec{y}}(\vec{x}_0, \vec{y}_0)$ not singular.

Then $\exists$ neigh U of $\vec{x}_0$, V of $\vec{y}_0$, $\exists\, g \in C^r$ $g : U \to V$ s.t.

$$\{ (x,y) \in U \times V : F(x,y) = \vec{0} \} = \{ (x, g(x)) \ \ x \in U \}$$

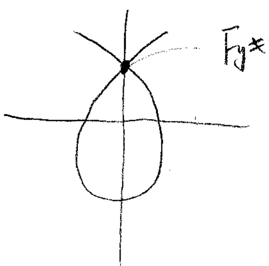
Immediate Application of the thm.

$e^{x^3} + e^y = 10$ $\qquad$ $F_y = e^y \neq 0$



Contradicts with Imp. Func. Thm.

$e^{x^2+y^2} + \sin xy = 5$



$F_y \neq 0$ This graph is not possible.