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Proof of Implicit function and Inverse function theorems

① Reducing Implicit function to inverse function

$$F : \underset{\underset{\mathbb{R}^n \times \mathbb{R}^m}{\cap}}{\Omega} \to \mathbb{R}^m$$

$$f : \Omega \to \mathbb{R}^{m+n} = \mathbb{R}^n \times \mathbb{R}^m$$

$$f(\underset{\mathbb{R}^n}{\vec{x}}, \underset{\mathbb{R}^m}{\vec{y}}) = (\underbrace{\vec{x}}_{\mathbb{R}^n}, \underbrace{F(x,y)}_{\mathbb{R}^m})$$

f' has following block structure:

$$f' = \underbrace{n\left\{\left(\begin{array}{c|c} \overbrace{I}^{n} & 0 \\ \hline F_{\vec{x}} & F_y \end{array}\right)\right\} n+m}_{n+m}$$

f satisfies assumption of inverse function theorem

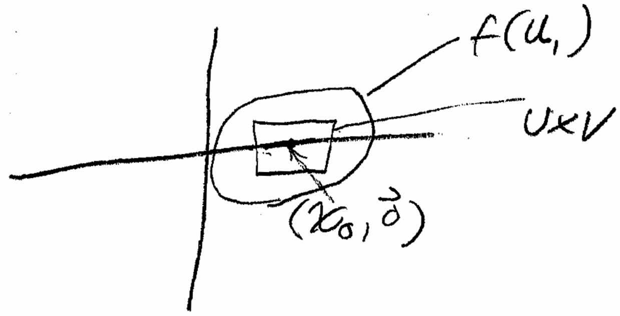
- Continuously differentiable
- Have to check $f'(\vec{x}_0, \vec{y}_0)$ not singular.
  (True because block triangular matrix with non-singular blocks).

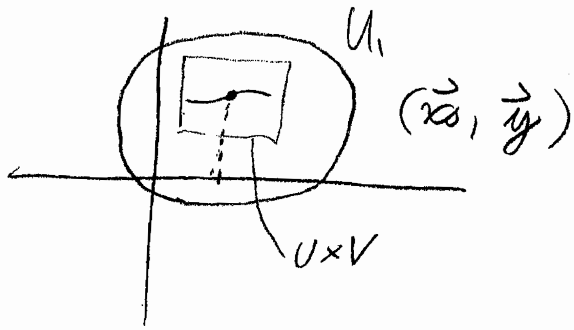
$f(\vec{x}_0, \vec{y}_0) = (\vec{x}_0, 0)$

g inverse for f.

$\exists$ neighborhood U_1 of $\vec{x}_0, \vec{y}_0$ s.t. $f(U_1)$ open.

$f: U_1 \to f(U_1)$ homeomorphism and





$\cancel{(x, g(x))} = \underbrace{G(\vec{x}, \vec{0})} = (x, g(x))$ we get g from implicit fn theorem

This is how implicit function theorem is reduced to inverse function theorem.

$f(g(x), x) = 0$ Condition that g is inverse for f.

(2) Contractive mapping principle

Thm X, ρ complete metric space

$\varphi: X \to X$ is a strict contractive mapping, i.e. $\exists\, r < 1$ s.t

$\rho(\varphi(x), \varphi(y)) < r \cdot \rho(x, y) \ \forall x, y \in X$.

Then $\exists!\ x_* \in X$ s.t. $f(x_*) = x_*$ (fixed point)

Moreover, $\forall x_0 \in X$

$x_* = \lim_{n \to \infty} x_n, \ x_n = \varphi(x_{n-1})$

Pf $\varphi(x_*) = x_*$

$\varphi(y_*) = y_*$

$$\rho(\varphi(x_*)\ \varphi(y_*)) \le r\rho(x_*\ y_*)$$

$$\begin{array}{c} \parallel \\ \rho(x_*, y_*) \end{array}$$

$$1 \le r \Rightarrow\Leftarrow$$

$$\Rightarrow \rho(x_*, y_*) = 0 \Rightarrow x_* = y_*$$

Existence

$$\rho(x_0\ x_1) = \rho(x_0\ \varphi(x_0)) = d$$

$$\rho(x_1\ x_2) = \rho(x\ \varphi(x_1)) = \rho(\varphi(x_0)\ \varphi(x_1)) \le rd$$

$$\rho(x_2\ x_3) = \rho(\varphi(x_1),\ \varphi(x_2)) \le r \cdot \rho(x_1\ x_2) \le r^2 d$$

$$\ldots$$

$$\rho(x_k\ x_{k+1}) \le r^k \cdot d$$

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for $m > n$ $\quad \rho(x_n\ x_m) \le \sum_{k=n}^{m-1} \rho(x_k\ x_{k+1})$

$$\le \sum_{k=n}^{\infty} r^k d = \frac{r^k d}{1-r} \xrightarrow[n\to\infty]{} 0$$

x_n - Cauchy seq.

$$\Rightarrow \exists \lim x_n = x_*$$

$$\varphi(x_*) = \lim_{n\to\infty} \varphi(x_n)$$

$$= \lim_{n\to\infty} x_{n+1} = x_*$$

Used continuity of φ

Proof of Inverse function Theorem

$\boxed{\begin{array}{l}\text{WLOG} \\ 1.\ \vec{x}_0 = \vec{0} \quad \vec{y}_0 = 0 \\ 2.\ f'(\vec{0}) = I \\ \quad \text{Repl. } f \text{ by } [f'(0)]^{-1} f\end{array}}$

$f(\vec{x}) = \vec{y}$

$0 = \vec{y} - f(\vec{x})$

$\vec{x} = \underbrace{\vec{y} + \vec{x} - f(\vec{x})}_{\varphi_y(\vec{x})}$

$f'(\vec{0}) = I$

$\exists \delta > 0$ s.t. $\forall \vec{x}$ $\|\vec{x}\| < \delta \Rightarrow \|I - f'(x)\| < \frac{1}{2}$

$\forall \vec{x}, \vec{x}',$ $\|\vec{x}\|, \|\vec{x}'\| < \delta$

$\|\varphi_y(\vec{x}) - \varphi_y(\vec{x}')\| \le \frac{1}{2} \|x - \vec{x}'\|$ $(*)$

because $\|(\varphi_y)'(\vec{x})\| \le \frac{1}{2}$

$I - f'(x)$

Need to get a mapping from the set into itself.

$\cancel{\varphi : B_{y,\varepsilon} \to B_{y,\varepsilon}}$

$\cancel{B_{y,\varepsilon} \subset B_{0,\delta}}$

~~Follows immediately from the estimate (*).~~

$\varepsilon < \frac{\delta}{2}$

$\|y\| < \frac{\varepsilon}{2} \quad \varphi_y : B_{0,\varepsilon} \to B_{0,\varepsilon}$

$\|x\| < \varepsilon$

$\|\varphi_y(\vec{x}) - \varphi_0(\vec{x})\| = \|\vec{y}\| < \frac{\varepsilon}{2}$

$\|\varphi_0(\vec{x}) - \underbrace{\varphi_0(\vec{0})}_{\vec{0}}\| \leq \frac{1}{2}\|\vec{x} - \vec{0}\| < \frac{\varepsilon}{2}$

$\|\varphi_y(\vec{x}) - 0\| < \varepsilon$

$f(\vec{x}) = \vec{y}$