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10/31

Remainder of Inverse Function Thm Pf

$y < \varepsilon/2$

$\varphi_y : B_{0,\varepsilon} \to B_{0,\varepsilon}$

So φ_y has unique fixed point in $B_{0,\varepsilon}$

So $x = x - f(x) + y$ has unique solution in $B_{0,\varepsilon}$

$\forall y \in B_{0,\varepsilon/2} \qquad \exists!\ x \in B_{0,\varepsilon} \quad f(x) = y$

$f : f^{-1}(B_{0,\varepsilon/2}) \longrightarrow B_{0,\varepsilon/2}$

bijection since we have unique solution

$g : B_{0,\varepsilon/2} \longrightarrow f^{-1}(B_{0,\varepsilon/2}) \qquad g = f^{-1}$

Now we must prove that g is differentiable at 0.

$$\frac{g(\vec{y}) - \vec{y}}{\|\vec{y}\|} \xrightarrow[\|\vec{y}\| \to 0]{} 0$$

$\vec{y} = f(\vec{x}) \qquad \vec{x} = g(\vec{y})$

$g(\vec{y}) - \vec{y} = \vec{x} - f(\vec{x})$

We know that $\dfrac{f(\vec{x}) - \vec{x}}{\|\vec{x}\|} \xrightarrow[\|\vec{x}\| \to 0]{} 0$

$\|\vec{x} - f(\vec{x})\| \leq \frac{1}{2}\|\vec{x}\| \qquad$ if $\|\vec{x}\| \leq \varepsilon$

$\Downarrow$

$\|f(\vec{x})\| \geq \frac{1}{2}\|\vec{x}\|$

$$\frac{g(\vec{y}) - \vec{y}}{\|\vec{y}\|} = \frac{\vec{x} - f(\vec{x})}{\|\vec{x}\|} \cdot \underbrace{\frac{\|\vec{x}\|}{\|\vec{y}\|}}_{\leq 2}$$

$\|\vec{y}\| \to 0 \Rightarrow \|\vec{x}\| \to 0$

$\to 0$ as $\|\vec{y}\| \to 0 \qquad \Rightarrow$ Inverse function is Differentiable

Let us prove $g \in C^r$, given g is differentiable as just proved

$$g(f(\vec{x})) = \vec{x}$$
$$g'(\vec{y})\, f'(\vec{x}) = I$$
$$g'(\vec{y}) = [f'(x)]^{-1} \quad \text{where } \vec{y} = f(\vec{x})$$

Let $A \in Mat_{n \times n}$ entries $\{a_{ij}\}_{i,j=1}^{n}$

Then entries of A^{-1} are rational functions of a_{ij} ($\det(A)$ is denominator)

C^∞ dependence on $\{a_{ij}\}$

$f \in C^r \Leftrightarrow f' \in C^{r-1}$

Note: Error from previous lecture

$(\vec{x}, g(\vec{x})) = G(\vec{x}, 0)$, not $g(\vec{x}) = G(\vec{x}, 0)$

Manifolds in $\mathbb{R}^N$

Def: $S \subset \mathbb{R}^N$ is called C^r manifold of dimension n if $\forall x_0 \in S$ $\exists$ open neigh $V \ni x_0$ in $\mathbb{R}^N$ and injective

$\varphi : \underset{\mathbb{R}^n}{\underset{\cap}{U}} \to \mathbb{R}^N$ $\quad \varphi \in C^r$

rank $\varphi'(\vec{u}_0) = n$ where $\varphi(\vec{u}_0) = \vec{x}_0$

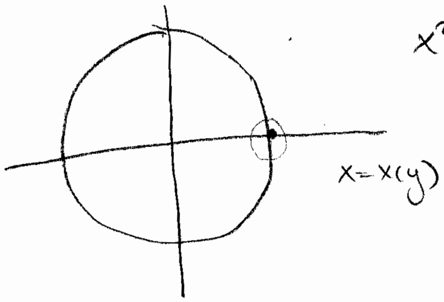
s.t. $S \cap V = \{ \varphi(\vec{u}) : u \in U \}$

Ex: Graph of $f : \mathbb{R}^n \to \mathbb{R}^m$ $\quad m = N - n$

is a C^r manifold manifold of dim n if $f \in C^r$

$(\underset{\mathbb{R}^n}{\underset{\cap}{\vec{x}}}, \underset{\mathbb{R}^m}{\underset{\cap}{f(\vec{x})}})$ $\quad \varphi(\vec{x}) = \begin{pmatrix} \vec{x} \\ f(\vec{x}) \end{pmatrix}$ then $\varphi'(\vec{x}) = \begin{pmatrix} I_{n \times n} \\ f'(x) \end{pmatrix}$ ← Invertible $\Rightarrow$ full rank n

Ex

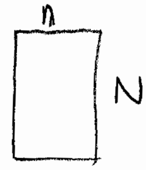


$x^2+y^2=1$

Informal: A manifold can be locally represented as a graph.

Pf (Before formal statement): $\varphi : \underset{\mathbb{R}^n}{\overset{\cap}{U}} \to \mathbb{R}^N \qquad \varphi(\vec{u}_0) = \vec{x}_0$

rank $\varphi'(\vec{u}_0) = n$

$\varphi' \to$ 

we can find n rows such that these rows form an $n \times n$ invertible matrix

$$\mathbb{R}^N = \underset{\vec{x}}{\mathbb{R}^n} \times \underset{\vec{y}}{\mathbb{R}^{N-n}}$$

$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad \varphi_1'(\vec{u}_0)$ is invertible

$\psi = \varphi_1^{-1}$

$\vec{x} = \varphi_1(\vec{u})$

$\vec{u} = \psi(\vec{x})$

$$\begin{pmatrix} \varphi_1(\vec{u}) \\ \varphi_2(\vec{u}) \end{pmatrix} = \underbrace{\begin{pmatrix} \varphi_1(\psi(\vec{x})) \\ \varphi_2(\psi(\vec{x})) \end{pmatrix}}_{\text{graph}} = \underbrace{\begin{pmatrix} x \\ \varphi_2(\psi(x)) \end{pmatrix}}$$

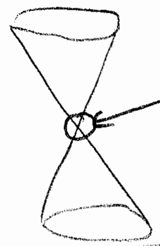
Thm: Let $S \subset \mathbb{R}^N$ be such that $\forall \vec{x}_0 \in S$ $\exists$ V neigh of x_0 and C^r function $F : V \to \mathbb{R}^{N-n}$ s.t. $F'(\vec{x}_0)$ has rank $N-n$ and $S \cap V = \{\vec{x} : F(\vec{x}) = \vec{0}\} \cap V$ Then S is a manifold C^r of dim n.

Ex 1: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $\quad a, b, c \neq 0$ is a manifold

$$F' = \left(\frac{2x}{a}, \frac{2y}{b}, \frac{2z}{c}\right)$$

$\neq 0$ on S,

Ex 2: $z^2 = x^2 + y^2$ – a double cone. This is not a manifold



because this neighborhood cannot be representsed as a graph.

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2 ways of defining manifolds (both local)

① Define local coordinates (function φ)

② Defining function F from Thm

Ex 3 $\quad SL(n) = \{ A \in M_{n \times n} : \det(A) = 1 \}$

Ex 4 $\quad SO(n) = \{ A : A^T A = I \}$ $\qquad$ HW

Both manifolds