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MA 113

Local Descriptions of Manifolds

Description I

$\varphi : U \to \mathbb{R}^N$, $U \subset \mathbb{R}^n$

Ex. Spherical coordinates

$z = \rho \cos\phi$
$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$

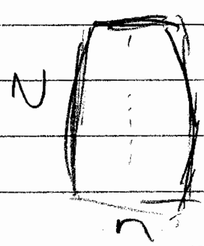
Local coordinates

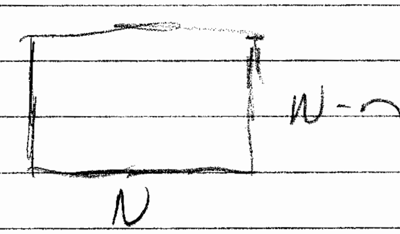
n-Manifolds can always be locally well-defined by n-coordinates

Description 2 - F defining function

$F(\vec{x}) = 0$ Ex. $x^2+y^2+z^2-R^2=0$

We always assume derivatives of φ and F have maximal rank (full rank)

$\varphi' =$ 

$F' =$ 

A manifold can locally be represented as a graph.

meaning given any neighborhood of M, we can pick n-variables as $\vec{x}$, remaining $N-n$ variables as $\vec{y}$, $S \cap U =$ graph of $\vec{y} = f(\vec{x})$

Let manifold S be represented by local coordinates φ.

Then we can find defining function f

Proof

Local Coordinate

$$\varphi : U \to \mathbb{R}^N \qquad \vec{x}_0 \in S$$

$$U \subset \mathbb{R}^n \qquad \vec{x}_0 \in V$$

$S \cap V = \varphi(U)$ and φ' has full rank

$\varphi(\vec{u}_0) = \vec{x}_0$

$$\varphi'(x_0) = \begin{array}{cc} \begin{bmatrix} \quad & \\ & \quad \end{bmatrix} & N \\ n & \end{array}$$

Introduce new matrix A $N \times (N-n)$ s.t.

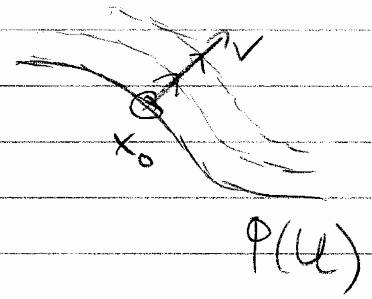
$\left[\varphi'(\vec{x}_0) \mid A \right]$ invertible

Define $\Phi : W \to \mathbb{R}^n$ neigh $(\vec{u}_0, \vec{0}) \in \mathbb{R}^N$

$\Phi(\vec{u}, \vec{v}) = \varphi(\vec{u}) + A(\vec{v})$

By inverse function theorem, Φ is invertible in neighborhood of $(\vec{u}_0, \vec{0}) \in \mathbb{R}^N$

Further explanation of previous proof



Inverse function theorem guarantees in local neighborhood of $\vec{x}_0$ no problems occur with transformation of curvilinear coordinates.

$\Psi = \varphi^{-1}$ is defined in a neighborhood of $\vec{x}_0$

both N-coordinates

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \begin{matrix} \text{— first } n\text{-coordinates } \vec{u} \\ \text{— remaining } N-n \text{ - coordinates } \vec{v} \end{matrix}$$

Claim
Ψ_2 is defining function

Manifold S $\quad \vec{x}_0 \in V$

$$S \cap V = \varphi_1(\mathcal{U}) = \varphi_2(\mathcal{U}) \qquad \varphi_1 \in C^r \quad \varphi_2 \in C^r$$

We can define $\underbrace{\varphi_2^{-1} \circ \varphi_1}_{\text{change of coordinate transformation.}} : \mathcal{U}_1 \to \mathcal{U}_2$

bijection

Thm: $\varphi_2^{-1} \circ \varphi_1 \in C^r$

Definition of Tangent Space to a manifold at point $\vec{x}_0$

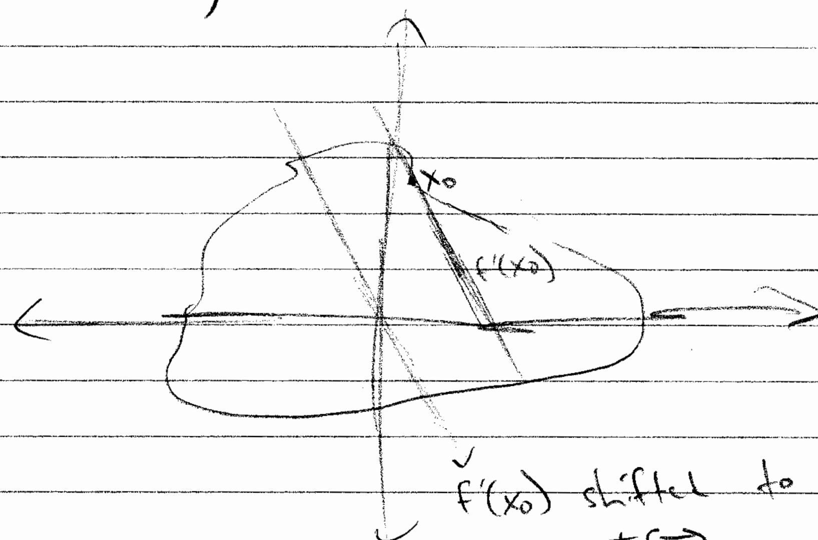
Notation

$$TS_{\vec{x}_0} = \left\{ f'(0) : \begin{array}{l} f(-\varepsilon, \varepsilon) \to S \\ f(0) = \vec{x}_0 \\ f \in C^1 \end{array} \right\}$$

$$f'(0) \in \mathbb{R}^n$$

$TS_{\vec{x}_0}$ is set of all possible velocities that go through 0.

Ex



Thm $TS_{\vec{x}} = Ran(\varphi'(\vec{u}_0)) = \ker F'(\vec{x}_0)$

Where φ is local coordinate, $\varphi(u_0) = x_0$ and F is defining function.

(No proof given - can be done using implicit function thm

Ex from HW

1) Manifold $S = SL(n)$
$n \times n$ matrices with determinant 1.

2) $S = O(n) = \{A \in M_{n \times n} : A^T A = I\}$

Find tangent spaces at point identity

Find derivatives and matrices that annihilate derivative.

These objects are Lee Groups

What is a Lie Group?

1) Manifold

2) Closed under multiplication / Addition
Group operations.

Names of these tangent spaces? Lie Algebras