Measure and Integration

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(To integrate bad functions)

ex $f(x) = \lim_{n\to\infty}\lim_{m\to\infty}\left(1-\cos^m(2\pi n!x)\right)$

$$= \begin{cases} 1 \text{ if } x \notin \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{Q} \end{cases}$$

$\int_0^1 f(x)dx =$ "length" of $\mathbb{Q}^c \cap [0,1]$

The set does not have any intervals in it, so you can't add it up normally.

Measure: $\Sigma \to \overline{\mathbb{R}}$ extended real line $- \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

$\Sigma \subset 2^X$

function of the set on X.

Def

$f: \Sigma \to \overline{\mathbb{R}}$, $\Sigma \subset 2^X$ is called additive if $\forall A_1, A_2, A_n \in \Sigma$

s.t. $A_i \cap A_j = \emptyset$ for all $i \neq j$

$\mu\left(\bigcup_1^n A_k\right) = \sum \mu(A_k)$ if $\bigcup_1^n A_k \in \Sigma$.

Countably additive if $\forall$ seq. of sets $A_k \in \Sigma$ s.t. $A_i \cap A_j = \emptyset$ if $i \neq j$

$\mu\left(\bigcup_1^\infty A_k\right) = \sum_{k=1}^\infty \mu(A_k)$ provided that $\bigcup_1^\infty A_k \in \Sigma$

ex $\Sigma = (a,b]$, $a<b \in \mathbb{R}$ (intervals)

$\mu((a,b]) = b-a$

μ is additive and countably additive.

2 disjoint sets, their total area is the area of each added. Need countably additive to do any analysis.

Def $\mathcal{A} \subset 2^X$ is called algebra if ($\emptyset \in \mathcal{A}$)

1. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
2. $A_1, A_2 \cdots A_n \in \mathcal{A} \Rightarrow \bigcup_1^n A_k \in \mathcal{A}$

σ-algebra if $\emptyset \in \mathcal{A}$

1. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
2. $A_k \in \mathcal{A}\ \forall k \in \mathbb{N} \Rightarrow \bigcup_1^\infty A_k \in \mathcal{A}$.

Remark

σ-algebra is algebra.

If σ-algebra, you don't have to worry about the condition $\bigcup_1^\infty A_k \in \mathcal{E}$ in the definition of countably additive. It is already satisfied

$A_k \in \mathcal{A}$ – σ-algebra

$\bigcap_{k=1}^\infty A_k = \left(\bigcup_{k=1}^\infty A_k^c\right)^c \in \mathcal{A}$

$A_k^c \in \mathcal{A}$

$\Rightarrow \bigcup_1^\infty A_k^c \in \mathcal{A}$

$\Rightarrow$ by ① indefinition

σ-algebra – countable unions, intersections, complements ok.

Def A measure on X is a countably additive non-negative $\mu: \mathcal{A} \to \overline{\mathbb{R}}$

$\mathcal{A}$ – σ-algebra $\subset 2^X$

ex $\mathcal{A} = 2^X \quad a \in X$

$\delta_a : 2^X \to \overline{\mathbb{R}}$ — countably additive measure.

$$\delta_a(A) = \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases}$$

ex $\mu : 2^X \to \overline{\mathbb{R}}$

$\mu(A) =$ "card(A)" ($+\infty$ for infinite sets) – is a measure

– called counting measure.

General problem

We have some premeasure $\mu_0 : \mathcal{E} \to \overline{\mathbb{R}}$. $\quad \mathcal{E} \subset 2^X$

Want to find a measure $\mu : \underset{\sigma\text{-algebra}}{\mathcal{A}} \to \overline{\mathbb{R}}$, $\mathcal{A} \supset \mathcal{E}$

$\mu_0(A) = \mu(A) \ \forall A \in \mathcal{E}$. Can do this.

Def $\mathcal{E} \subset 2^X$ is called an elementary family (semiring) if $\emptyset \in \mathcal{E}$ and

① $A, B \in \mathcal{E} \Rightarrow A \cap B \in \mathcal{E}$

② $A \in \mathcal{E} \Rightarrow A^c = \bigcup_{k=1}^{n} A_k$

$A_k \in \mathcal{E}$, $A_k \cap A_j = \emptyset$ if $k \neq j$

ex ① Intervals,
② Rectangles in $\mathbb{R}^2$
③ Parallelpiped in $\mathbb{R}^3$

① $(a, b]$ $a, b \in \overline{\mathbb{R}}$, $a \leq b$

allow (a, ∞) and $(-\infty, b]$

$(a,b]^c = (-\infty, a] \cup (b, \infty)$

Def $\mathcal{E} \subset 2^X$ A sigma algebra (or algebra) generated by $\mathcal{E}$ is the smallest σ-alg (or alg) $\mathcal{A} \supset \mathcal{E}$ — 2 definitions: for σ-alg and alg

Why it exists: Take all σ-algebras (or algebras) $\supset \mathcal{E}$ and $\bigcap$. take intersection

2^X – σ-alg, $2^X \supset \mathcal{E}$

Thm $\mathcal{E}$ – elementary family.
and $\mathcal{A}(\mathcal{E}) = \{$finite unions of pairwise disjoint $A_k \in \mathcal{E}\}$
Then $\mathcal{A}(\mathcal{E})$ is algebra generated by $\mathcal{E}$

Only need to prove $\mathcal{A}(\mathcal{E})$ is algebra. Because all algebras that contain $\mathcal{E}$ must contain all finite unions of pairwise elements of $\mathcal{E}$