11/7/07

<u>Notation</u> $\mathcal{A}$ - algebra
$\mathfrak{A}$ - σ-algebra
$\mathcal{A}(\mathcal{E})$, $\mathfrak{A}(\mathcal{E})$ - algebra/σ-algebra generated by $\mathcal{E}$

For elementary family $\mathcal{E}$,
$\mu_0 : \mathcal{E} \to [0,\infty]$ can be extended to $\mathcal{A}(\mathcal{E})$ as follows
$\underset{2^X}{\cap}$
$\mu_0\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu_0(E_k)$ if $E_k \cap E_j = \emptyset \ \forall j \neq k$

If μ_0 was additive, then its extension is well-defined and additive.
If μ_0 was countably additive, then μ is countably additive.

$A = \bigcup_1^n E_k = \bigcup_1^m F_j$
$\{E_k\}$ disjoint
$\{F_j\}$ disjoint
$E_{kj} = E_k \cap F_j$
$E_{k,j}$ are disjoint

$$\sum_{k=1}^{n} \sum_{j=1}^{m} \mu_0(E_{kj}) = \sum_{k=1}^{n} \mu_0\Big(\underbrace{\sum_{j=1}^{m} E_{kj}}_{E}\Big)$$
$$= \sum_{k=1}^{n} \mu_0(E_k)$$
different-order of summation matters
$$\sum_{j=1}^{m} \sum_{k=1}^{n} \mu_0(E_{kj}) = \sum_{j=1}^{m} \mu_0(F_j)$$

Situation: "Premeasure"
$\mu_0 : \mathcal{E} \to [0,\infty]$
μ_0 - countably additive
Assume extended μ_0 to $\mathcal{A}(\mathcal{E})$
Want to extend μ_0 to σ-algebra $\mathfrak{A}(\mathcal{E}) = \mathfrak{A}(\mathcal{A}(\mathcal{E}))$

<u>Prop</u> Length on intervals is countably additive

Take intervals of the form $(a,b]$ with $a<b$
$\ell((a,b]) = b-a$. $\ell(\emptyset) = 0$

Proof: $I = (a,b]$

$I = \bigcup_1^\infty I_k$

$I_k = (a_k, b_k]$

$I_k \cap I_j = \emptyset$ for $j \neq k$

1) $\bigcup_1^n I_k \subset I \Rightarrow \ell(\bigcup_1^n I_k) \leq \ell(I)$

$$\underset{\sum_1^n \ell(I_k)}{\overset{\shortparallel}{}}$$

$\lim_{n\to\infty}: \sum_1^\infty \ell(I_k) \leq \ell(I)$

2) $\varepsilon > 0$

$\tilde{I}_k = (a_k - 2^{-k}\varepsilon, b_k - 2^{-k}\varepsilon)$

$J_k = (a_k - \varepsilon 2^{-k}, b_k - \varepsilon 2^{-k}]$

$\bigcup_k \tilde{I}_k \supset I \supset [a+\varepsilon, b]$ $\qquad I = [a,b]$

$\exists$ finite subcover $\tilde{I}_{k_j}$ s.t.

$\bigcup_{j=1} \tilde{I}_{k_j} \supset [a+\varepsilon, b]$

$\Rightarrow \bigcup_{j=1}^n J_k \supset [a+\varepsilon, b]$

$\Rightarrow \sum_{j=1}^n \ell(J_k) \geq \ell(I) - \varepsilon$

$$\sum_{k=1}^\infty \underbrace{[\ell(I_k) + 2\varepsilon \cdot 2^{-k}]}_{\ell(J_k)} \geq \sum_{j=1}^N \underbrace{(\ell(I_{k_j}) + 2\varepsilon \cdot 2^{-k})}_{\ell(J_{k_j})}$$

$$\geq \ell(I) - \varepsilon$$

$\sum_{k=1}^\infty \ell(I_k) + 2\varepsilon \geq \ell(I) - \varepsilon$

$\sum_{k=1}^\infty \ell(I_k) \geq \ell(I) - \varepsilon \quad \forall \varepsilon > 0$

$\lim_{n\to\infty}: \sum_1^\infty \ell(I_k) \geq \ell(I)$

$\Rightarrow \sum_1^\infty \ell(I_k) = \ell(I)$

<u>Remark</u> If $\mu: \mathcal{A} \to [0,\infty]$ is additive, then μ is monotone.

$\hookrightarrow A \subset B \Rightarrow \mu(A) \leq \mu(B)$

Remark $\mu: \mathcal{A} \to [0,\infty]$ is countably additive iff $\forall$ seq. $A_n \in \mathcal{A}$ s.t. $A_n \subset A_{n+1}$ and $\bigcup_1^\infty A_n \in \mathcal{A}$, $\mu(\bigcup_1^\infty A_n) = \lim \mu(A_n)$

$B_n = A_n \setminus A_{n-1} \quad n \geq 2$

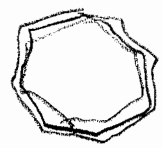
$B_1 = A_1$

$\bigcup_1^\infty B_n = \bigcup_1^\infty A_n$, B_n disjoint

B_n disjoint and $A_n = \bigcup_1^n B_k$

[Given an increasing family, disjoint sets can be created]

Jordan measure:



Approximate E by $E_0 \subset \mathcal{A}(\varepsilon)$ s.t. $E \Delta E_0 = (E \setminus E_0) \cup (E_0 \setminus E)$ "small"

Outer measure

μ_0 on $\mathcal{A} \subset 2^X$

$\forall A \subset X$, define $\mu^*(A) = \inf\{\sum_{k=1}^\infty \mu_0(A_k) : A_k \in \mathcal{A},\ \bigcup_1^\infty A_k \supset A\}$

μ^* is the outer measure

E is measurable if $\forall \varepsilon\ \exists E_\varepsilon \subset \mathcal{A}$ s.t. $\mu^*(E \Delta E_\varepsilon) < \varepsilon$
(informal definition of measurable)

Def ℓ_0 - length on intervals, ℓ^* - corresponding outer measure. If $\ell^*(E) = 0$, then E has Lebesgue measure 0.

Example $\mathbb{Q}$

$\ell^*(\mathbb{Q}) = \emptyset$