Premeasure
$\downarrow$
outer measure (way to measure "smallness")

1. You do not need premeasure to define outer measure.

<u>Lemma</u> $\lambda : \mathcal{C} \to [0, \infty]$ $\mathcal{C} \subset 2^X$
assume $\exists C_n \subset \mathcal{C}$ s.t. $\bigcup_1^\infty C_n = X$
Let $\mu^*(E) = \inf \{ \sum \lambda(C_k) : C_k \ \bigcup_1^\infty C_k \supset E \}$
$\mu^*(\emptyset) = 0$

Then μ^* is outer measure (i.e. μ^* is countably semiadditive)

Outer measure is a countably subadditive function
$2^X \to [0, \infty]$

Ex. Hausdorff measure
$H_{\delta, p}$: outer measure
$\lambda = (\text{diam } C)^p$
$\mathcal{C} = \{ C_{\subset \mathbb{R}^n} : \text{diam } C \leq \delta \}$

$H_{\delta,p}(E) := \inf \{ \sum (\text{diam } C_k)^p : \bigcup C_k \supset E, \text{diam } C_k \leq \delta \}$

$\delta' < \delta \Rightarrow H_{\delta', p}(E) \geq H_{\delta, p}(E) \Rightarrow \exists \lim_{\delta \to 0} H_{\delta, p}(E) =: H_p($

Hausdorff measure of dim p

Hausdorff measure is outer measure

<u>Lemma</u> Let μ_n - outer measure s.t. $\forall E$ $\mu_n(E) \nearrow$ $(\mu_n(E) \leq \mu_{n+1}(E))$
Then $\mu(E) = \lim_{n \to \infty} \mu_n(E)$ is an outer measure

2. One can get a measure from outer measure μ^* by restricting it to appropriate σ-algebra

<u>Def</u> μ^* - outer measure on X

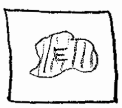
We say $E \subset X$ is μ^* measurable if $\forall A \subset X$,

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$

<u>Thm</u> (Caratheodory extension thm)

The set $\mathcal{M}$ of μ^* measurable functions is a σ-algebra and $\mu^* | \mathcal{M}$ is a measure

<u>Ex</u> μ_0, $\mu_0(X) < \infty$ premeasure



X - unit square, premeasure is area of rectangles

construct μ^* - outer measure from μ_0

we need $\mu^*(E) + \mu^*(X \cap E^c) = \mu^*(X) = \mu_0(X)$

can write either $\updownarrow$

$\mu^*(X \cap E)$ since $E \subset X$

definition of Lebesgue measurable sets

<u>Pf</u> of Caratheodory thm.

1. $\mathcal{M}$ is algebra

   a) $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$ trivial

   b) $E, F \in \mathcal{M} \overset{?}{\Rightarrow} E \cup F \in \mathcal{M}$

   Take arb $A \subset X$

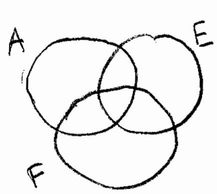
   $\mu^*(A) \underset{\uparrow E \in \mathcal{M}}{=} \mu^*(A \cap E) + \mu^*(A \cap E^c)$

   $= \mu^*(A \cap E) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c)$

   $F \in \mathcal{M}$ (use $A \cap E^c$ instead of A)

   $\geq \mu^*((A \cap E) \cup (A \cap E^c \cap F)) + \mu^*(A \cap E^c \cap F^c)$

   $= \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F^c))$



$(E \cup F)^c = E^c \cap F^c$

look at Venn Diagram to see this

Remark $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$
by countable subadditivity