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MATH 1130
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Caratheodory Theorem

Useful in general situation, hard to show for specific sets.

Only tells us that $\mathcal{M}$ - σ-algebra.

Does not tell us that $\mathcal{M} \supset \mathcal{A}$

Your σ-algebra could be trivial: $\{\emptyset, X\}$

~~Theorem~~

Lemma

μ_0 premeasure on $\mathcal{A}$

μ^* - outer measure constructed from μ_0

Then $\forall A \in \mathcal{A}$ $\mu^*(A) = \mu_0(A)$

Pf

$\mu^*(A) \leq \mu_0(A)$ is trivial:

$$\mu^*(A) = \inf\{\Sigma \mu_0(A_k) : A_k \in \mathcal{A}, \bigcup_1^\infty A_k \supset A\}$$

Need to prove $\mu^*(A) \geq \mu_0(A)$

Enough to consider $\mu^*(A) < \infty$

Take $\varepsilon > 0$ $\exists A_k \in A, \bigcup_k A_k \supset A$

$\Sigma \mu_0(A_k) < \mu^*(A) + \varepsilon$

$\tilde{A}_k$: $\tilde{A}_1 = A_1 \cap A$

$\tilde{A}_{k+1} = (A_{k+1} \setminus \bigcup_1^k A_j) \cap A$

$\bigcup_1^n \tilde{A}_k = (\bigcup_1^n A_k) \cap A$

$\bigcup_1^\infty \tilde{A}_k = A$

so $\sum_1^\infty \mu_0(\tilde{A}_k) \leq \sum_1^\infty \mu_0(A_k) \leq \mu^*(A) + \varepsilon$

$\tilde{A}_k \subset A_k$

$\mu_0(A)$

$\mu_0(A) \leq \mu^*(A) + \varepsilon \quad \forall \varepsilon > 0 \Rightarrow \mu_0(A) \leq \mu^*(A) \Rightarrow \mu_0(A) = \mu^*(A)$ $\square$

Lemma

μ_0 premeasure on $\mathcal{A}$

$\mu^*, \mathcal{M}$

Then $\mathcal{A} \subset \mathcal{M}$

Pf

$A \in \mathcal{A}$, $E \subset 2^X$

? $\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E)$

Pf

$\forall\, \varepsilon > 0\ \exists\, A_k \in \mathcal{A}$

$\bigcup_1^\infty A_k \supset E$, $\sum \mu_0(A_k) \le \mu^*(E) + \varepsilon$

$A \cap E \subset \bigcup_1^\infty (A_k \cap A)$

$A^c \cap E \subset \bigcup_1^\infty (A_k \cap A^c)$

$(A_k \cap A^c) \cap (A_k \cap A) = \phi$

$$\underbrace{\sum_1^\infty \mu_0(A_k \cap A^c)}_{\displaystyle \ge\ \mu^*(A^c \cap E)} + \underbrace{\sum_1^\infty \mu_0(A_k \cap A)}_{\displaystyle \ge\ \mu^*(A \cap E)}$$

$\mu^*(A \cap E) + \mu^*(A^c \cap E) \le \mu^*(E) + \varepsilon \quad \forall\, \varepsilon > 0$

$\Rightarrow \mu^*(A \cap E) + \mu^*(A^c \cap E) \le \mu^*(E)$

$\ge$ trivial (subadditivity of μ^*)

Thm μ_0 - Premeasure on $\mathcal{A}$, μ^* outer from μ_0

Then $\mathcal{M} \supset \mathcal{A}$ and $\mu^*|_{\mathcal{A}} = \mu_0$

Ex μ_0 length

$\mu_0((a, b]) = b - a$

m^* - Lebesgue outer measure

$m^*|_{\mathcal{M}}$ = Lebesgue measure

$\mathcal{M}$ - Lebesgue measurable sets

Ex Lebesgue - Stiltjess measure

$F \nearrow \quad \mathbb{R} \to \mathbb{R}$

$\mu_0((a,b]) = F(b_+) - F(a_+)$

where $f(x_+) = \lim_{s \to x+} F(s)$

Need to show that μ_0 is countably additive

$dF \ldots$

If $F \in C^1$

then $\mu(E) = \int_E F'(x)\,dx = \int_E dF$

Ex $F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

Then $dF = \delta_0$

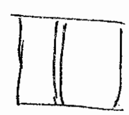
~~A premeasure~~

A premeasure μ_0 can have many extensions.

Ex

$X = [0,1] \times [0,1]$

$\mathcal{A}$ generated by $(a,b] \times [0,1]$



"strips" (vertical)

$\mu_0((a,b] \times [0,1]) = b - a$

Could say measure is area, or trace of area.

Thm

If $\mu_0(X) < \infty$, μ_0 on $\mathcal{A}$

Let $\mathcal{A} \subseteq \mathcal{O} \subset \mathcal{M}$

Let μ be an extension of μ_0 onto $\mathcal{O}$

Then $\mu(A) = \mu^*(A)\ \forall A \in \mathcal{O}$

* The only way to get non-uniqueness is to play with how you define the measure

*Caratheodory - unique "where it counts"