11/16/07 Gregory Thompson

Thm

If μ_0 is a premeasure on X ($\mu : \mathcal{A} \to [0,\infty]$)

$\mu_0(X) < \infty$ and μ extension of μ_0 onto σ

$\mathcal{A} \subseteq \sigma \subseteq \mathcal{M}$

$\Rightarrow$ for $\forall A \in \sigma$ $\mu(A) = \mu^*(A)$

Lemma μ_0 - premeasure $\mathcal{A} \to [0,\infty]$

μ - extension to σ $\mathcal{A} \subset \sigma \subset \mathcal{M}$

$\Rightarrow \forall A \in \sigma$ $\mu(A) \leq \mu^*$

(Did not assume $\mu_0(X) < \infty$)

Pf (Thm from Lemma)

$$\mu(A) + \mu(A^c) = \mu(X) = \mu_0(X)$$

inequalities $\rightarrow$ $\mu(A) \geq \mu^*(A)$, $\mu(A^c) \geq \mu^*(A^c)$

$$\mu^*(A) + \mu^*(A^c) = \underbrace{\mu^*(X) = \mu_0(X)}$$

$\Rightarrow \mu^*(A) = \mu(A)$

$\mu^*(A^c) = \mu(A^c)$

$A \in \mathcal{M}$

$\downarrow$

$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E)$

$\forall E \in 2^X$ Take $E = X$

because $X \in \mathcal{A}$

Pf of Lemma

Take arb $\epsilon > 0$

$\exists A_n \subset \mathcal{A}$ st

$\bigcup^{\infty} \supset A$ and $\sum \mu_0(A_n) < \mu^*(A) + \epsilon$

$$A \subset \bigcup_1^{\infty} A_n, \quad \mu(A) \leq \mu(\bigcup_1^{\infty} A_n) \leq \sum_1^{\infty} \mu(A_n) = \sum_1^{\infty} \mu_0(A_n) < \mu^*(A) + \epsilon$$

monotonicity (follows from additivity)

countable subadditivity of a measure

$\mu(A) \leq \mu^*(A) + \epsilon \quad \forall \epsilon > 0$

Def A measure (premeasure)
μ is called σ-Finite on X
if $\exists$ μ-measurable X_n
$\bigcup_1^\infty X_n = X$, $\mu(X_n) < \infty$

μ-measurable $\equiv$ belongs to σ-alg (alg.) where μ defined
x m on $\mathbb{R}$
m_n on $\mathbb{R}^n$

R — $X_n = (-n, n]$

Remark Without Loss of Generality $X_n \subset X_{n+1}$
Consider $\bigcup_1^n X_k$

Lemma μ is σ-finite, $X_n \subset X_{n+1}$ From def
$\Rightarrow \mu(E) = \lim_{n\to\infty} \mu(E \cap X_n)$

Corillary Thm about
uniqueness of extension
is true For σ-finite premeasures

Def A measure $\mu : \mathcal{O}\!\mathcal{C} \to [0,\infty]$ is called complete
if $\forall E \in \mathcal{O}\!\mathcal{C}$, $\mu(E) = 0$
$\Rightarrow 2^E \subset \mathcal{O}\!\mathcal{C}$ (any subset of E belongs to $\mathcal{O}\!\mathcal{C}$, measurable)

Trivial Observation
If μ_0, μ^* If $\mu^*(E) = 0$
$\Rightarrow$ E is μ^* measurable
$E \in \mathcal{M}$

Remark μ obtained From μ_0 by Caratheodory is complete

Def A Borel σ-alg on a topological space X is a σ algebra
Generated by $\forall$ open sets
same as σ-algebra generated by $\forall$ closed sets

Borel σ-alg on $\mathbb{R}$

$\mathcal{B}$ - Borel σ-alg

$\mathcal{B}_1 = \sigma$-alg generated by all $(-\infty, a]$

$\mathcal{B}_2 = \sigma$-alg generated by all intervals $(a, b]$ and all $(-\infty, a]$ and all (b, ∞)

Lemma $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$

$(-\infty, a)$, $[a, \infty)$ intervals will be same

Pf Lemma $\mathcal{B}_1 \subset \mathcal{B}_2$ trivial

$(a, b] = (-\infty, b] \cap (-\infty, a]^c$

$\Rightarrow (a, b] \in \mathcal{B}_1$ $(a, \infty) = (-\infty, a]^c$

$\Rightarrow \mathcal{B}_2 \subset \sigma(\mathcal{B}_1) = \mathcal{B}_1$

so $\mathcal{B}_1 \subset \mathcal{B}_2$ and $\mathcal{B}_2 \subset \mathcal{B}_1 \Rightarrow \mathcal{B}_1 = \mathcal{B}_2$

$(-\infty, a]$ - closed so $\mathcal{B}_1 \subset \mathcal{B}$

$(a, b) = \bigcup_1^\infty (a, b - \frac{1}{n}]$

so $(a, b) \in \mathcal{B}_1$

any open $U = \bigcup_1^\infty (a_n, b_n)$

$\Rightarrow U \in \mathcal{B}_1$

$\Rightarrow \mathcal{B} \subset \sigma(\mathcal{B}_1) = \mathcal{B}_1$

so $\mathcal{B} = \mathcal{B}_1$

Thm Any open $U \subset \mathbb{R}$ is a countable union of disjoint open intervals

---

(a, b) $a, b \in \mathbb{Q}$ is the Base of topology in $\mathbb{R}$

But card $\{(a, b) : a \in \mathbb{Q}\ b \in \mathbb{Q}\} = \#\mathbb{N}$