UNIQUENESS OF TRANSLATION INVARIANT BOREL MEASURES

Def measure μ - Borel measure on $\mathbb{R}^n$ is called translation invariant

if $\forall\ E \in \mathcal{B}, \forall \vec{x} \in \mathbb{R}^n$,

$$\mu(x+E) = \mu(E)$$

$x+E$ is translation of all points in E by x.

$$x+E = \{x+y : y \in E\}$$

Lebesgue measure of dim n,

m_n is translation invariant.

Turns out that this is the only one.

Thm Let μ be a translation invariant Borel measure, s.t. $\mu(P) < \infty$ for some parallelipiped P, $\text{int}(P) \neq \emptyset$

(i.e. no dimensions have length 0)

Then $\exists\, c$ s.t. $\mu(E) = c\, m_n(E)\ \forall E \in \mathcal{B}$

Proof (dim 1 - dim n is similar)

Let $\mu(I) < \infty$ for some $I = \langle a, b \rangle$ $a < b$ — does not matter if endpoints included or not

then $\mu((0,1])$ is also finite because $(0,1]$ can be covered by finitely many translations of $\langle a, b\rangle$

Then $\exists\, c$ s.t. $\mu((0,1]) = c\, \underbrace{m_1((0,1])}_{1}$

$\mu((0, 1/n]) = \frac{1}{n} c\, m_1((0,1])$ because $(0,1] = \bigcup_{k=0}^{n-1} \left(\frac{k}{n}, \frac{k+1}{n}\right]$ ← disjoint translations of same interval

$\mu((0, k/n]) = \frac{kc}{n}$ $\qquad = \bigcup_{k=0}^{n-1} \frac{k}{n}\left(0, \frac{1}{n}\right]$

$\Rightarrow \mu((a,b]) = c\, m_1((a,b])$

for $a, b \in \mathbb{Q}$

$\Rightarrow \mu((a,b]) = c\, m_1((a,b])$

$\forall a, b \in \mathbb{R}$ → approximate arbitrary interval by rational interval

APPROXIMATION

$\lim a_n = \lim \tilde{a}_n = a$

$\lim b_n = \lim \tilde{b}_n = b$

$a_n, b_n, \tilde{a}_n, \tilde{b}_n \in \mathbb{Q}$

$\mu = \mu^*|_{\mathcal{B}} = c\,m_1^*|_{\mathcal{B}} = c\,m_1|_{\mathcal{B}}$ Caratheodory Extension

Since $\mathcal{B} \subset \mathcal{M}$, Caratheodory Extension is unique.

LINGO

Radon measure, Regular Borel measure on $\mathbb{R}^n$ means Borel measure s.t.

$\mu(P) < \infty$ $\forall$ bounded parallelipiped $P \subset \mathbb{R}^n$

HAUSDORFF MEASURES

$H_{p,\delta}(E) = \inf \{ \sum (\operatorname{diam} C_k)^p : E \subset \bigcup C_k,\ \operatorname{diam} C_k < \delta \}$

outer measure

$H_p(E) = \lim_{\delta \to 0} H_{p,\delta}(E)$ Hausdorff measure is outer measure.

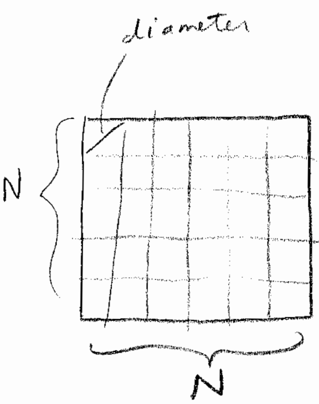
$\boxed{\text{Thm } H_p|_{\mathcal{B}} \text{ (in } \mathbb{R}^n) \text{ is countably additive.}}$

Hausdorff measure H_p is translation invariant.

Thm in $\mathbb{R}^n$

$H_n(E) = c(n) \cdot m_n(E)$ $\forall$ Borel set $E \subset \mathbb{R}^n$

$(0 < c(n) < \infty)$



PROOF enough to prove $0 < H_n((0,1]^n) < \infty$

split cube into N^n cubes of diam $\sqrt{n} \cdot \frac{1}{N}$

let N be st. $\frac{1}{N}\sqrt{n} < \delta$ then

$$H_{p,\delta}((0,1]^n) \le \sum \left(\frac{1}{N}\sqrt{n}\right)^n = N^n \left(\frac{1}{N}\sqrt{n}\right)^n = n^{n/2}$$

Take $\lim_{\delta \to 0} \Rightarrow H_n((0,1]^n) \le n^{n/2} < \infty$ Now have to show $H_n > 0$

$C_K \subset B_K$, $\operatorname{diam}(B_K) = 2 \operatorname{diam} C_K$

$C_K \subset B_K$ where $\text{diam } B_K = 2 \text{ diam } C_K$

$\bigcup B_K \supset E$

If $\text{diam } C_K < \delta$ then $\text{diam } B_K < 2\delta$

$$\sum (\text{diam } B_K)^n = 2^n \sum_k (\text{diam } C_k)^n$$

also,

$$\sum (\text{diam } B_K)^n = \sum_k v_n \, m_n(B_K) \quad \exists\, v_n \quad \left(\begin{array}{l}\text{ie. volume of } B_K \\ \propto (\text{diam } B_K)^n\end{array}\right)$$

$$\Rightarrow 2^n \sum_k (\text{diam } C_k)^n \geq v_n \sum_K m_n(B_K) \geq v_n \, m_n\left(((0,1])^n\right) = v_n$$

$$\Rightarrow \quad H_{p,\delta}\left((0,1]^n\right) \geq \frac{v_n}{2^n} \Rightarrow H_p\left((0,1]^n\right) \geq \frac{v_n}{2^n} > 0.$$

So, $H_n = c(n)\, m_n$ and $0 < c(n) < \infty$

<u>Corollary</u> m_n in $\mathbb{R}^n$ is rotationally invariant.

Q. $H_p\left((0,1]^n\right)$ for all p ?