- Hausdorff dimension

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<u>Prop:</u> $K \subset X$ metric

1) if $H_p(K) < \infty \Rightarrow H_r(K) = 0 \quad \forall r > p$
2) if $H_p(K) > 0 \Rightarrow H_r(K) = \infty \quad \forall r < p$

including ∞

<u>Idea of Proof</u> (Key step). Full <u>PF</u> left for exercise.

if $\operatorname{diam} C_k < \delta$ then $(\operatorname{diam} C_k)^r \leq (\operatorname{diam} C_k)^p \delta^{r-p} \quad r > p$

$\geq (\operatorname{diam} C_k)^p \delta^{r-p} \quad r < p$

<u>2nd Idea of Proof</u>

If $H_p(K) < \infty \Rightarrow H_{p,\delta}(K) < \infty \quad \forall$ small δ

<u>Cor:</u> $K \subset \mathbb{R}^n \quad \exists!\, p_0 \in [0, n]$ st $H_p(K) = 0 \quad \forall p > p_0$

$H_p(K) = \infty \quad \forall p < p_0$

$\boxed{p_0 \text{ is called Hausdorff dim of } K}$

$$P_0 = \inf\{\underbrace{p : H_p(K) = 0}_{[b, \infty)}\} = \sup\{\underbrace{p : H_p(K) = +\infty}_{(0, a)}\}$$

$p \in (a, b)$ $0 < H_p(K) < \infty$ but this is impossible

$\Rightarrow a = b \Rightarrow p_0 = a = b$

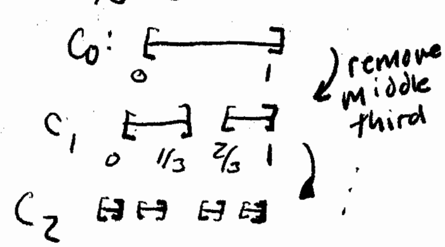
Examples: 1) $I = (a, b)$

$H\text{-dim}(I) = 1$

2) $Q = (0, 1]^n$ in $\mathbb{R}^n$

$H\text{-dim}\, Q = n$

3) 1/3 Cantor set



$\vdots$

$C = \bigcap_{1}^{\infty} C_n$

$H\text{-dim}\, C = \dfrac{\ln 2}{\ln 3}$

Heuristics

Let for some p

$0 < H_p(C) < \infty$

$H_p(C) = H_p(C_+) + H_p(C_-)$ where $C_+ = C \cap (1/2, 1]$, $C_- = C \cap [0, 1/2]$

$= 2 H_p(C_+)$

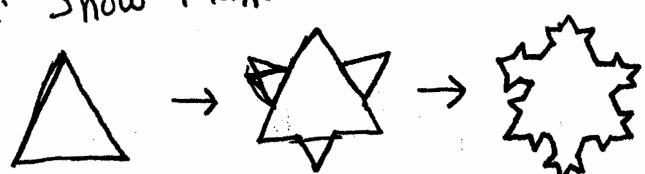
C obtained from $3C$ by translation (self-similar)

$= 2 H_p(C) \cdot \left(\frac{1}{3}\right)^p$

$1 = \frac{2}{3^p} \Rightarrow p = \frac{\ln 2}{\ln 3}$

Remark: $\exists$ sets in $\mathbb{R}^2$ st H-dim $K = 1$ but $H_1(K) = \infty$ (Think of something with ∞ length)

Another Example: Snow Flake



– replace middle 1/2 with another triangle, etc.

H-dim will be fractional.

Metric outer measure

def: Outer measure μ^* on metric space X is called metric outer measure if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$

$\forall A, B$ st $\text{dist}(A,B) = \inf\{\rho(x,y) : x \in A, y \in B\} > 0$

A, B called well-sepa

Metric Outer Measure (cont.)

μ^* can construct $\mathcal{M}$ of μ^* measurable

Thm: if μ^* - metric outer measure,
then $\mathcal{B} \subset \mathcal{M}$

Borel set

Prop: H_p is a metric outer measure

Pf: Let A, B $\text{dist}(A,B) = \delta > 0$

Need to show $H_p(A \cup B) = H_p(A) + H_p(B)$

$\le$ trivial - countable sub-additivity

need $\geq$

$\forall \varepsilon > 0 \ \exists \delta_0 > 0$ st
$\forall \delta < \delta_0 \quad H_{p,\delta}(A \cup B) < H_p(A \cup B) + \varepsilon/2$

Fix $\delta < \min(\delta_0, d)$ Find cover C_k, diam $C_k < \delta$

$$\bigcup C_k \supset A \cup B$$

st. $\sum (\text{diam } C_k)^p < H_{p,\delta}(A \cup B) + \varepsilon/2$
$< H_p(A \cup B) + \varepsilon$

For each k only 3 possibilities.

$A \cap C_k \neq 0 \quad I_1$

or $B \cap C_k \neq 0 \quad I_2$

or $C_k \cap (A \cup B) = \emptyset$

$$\sum (\text{diam } C_k)^p \geq \sum_{I_1} (\text{diam } C_k)^p + \sum_{I_2} (\text{diam } C_k)^p$$
$$\geq H_p(A) + H_p(B)$$

$H_p(A \cup B) + \varepsilon \geq H_p(A) + H_p(B) \quad \forall \varepsilon > 0 \implies H_p(A \cup B) \geq H_p(A) + H_p(B)$

END NOTES.