Scott Zager

Proof of theorem about metric outer measures:

Prove: $\mathcal{B} \subset \mathcal{M}$ (Borel σ-algebra)

· sufficient to show any closed set $F \in \mathcal{M}$.

need $\forall A \subset X$, $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c)$ [closed F]

Define $E_n = \{x \in A : dist(x, F) > \frac{1}{2^n}\}$

$A \cap F^c = \{x \in A : dist(x, F) > 0\}$, since F is closed

$= \bigcup_{n=1}^{\infty} E_n$

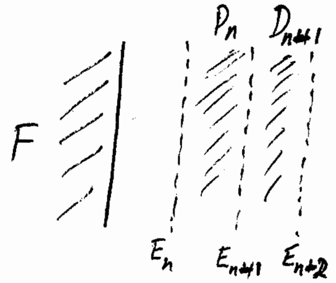
• Trivially, $\mu^*(A) \le \mu^*(A \cap F) + \mu^*(A \cap F^c)$

F and E_n are well-separated $\Rightarrow \mu^*(A \cap F \cup E_n) = \mu^*(A \cap F) + \mu^*(E_n)$

since μ^* is a metric outer measure and $\mu^*(A) \ge \mu^*(A \cap F \cup E_n)$

Goal: $\mu^*(E_n) \to \mu^*(A \cap F^c)$

let $D_n = E_{n+1} \setminus E_n$



· set D_{2n} are well-separated

– same with D_{2n+1}

$$\sum_{n=1}^{N} \mu^*(D_{2n}) = \mu^*\left(\bigcup_{n=1}^{N} D_n\right) \le \mu^*(A)$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \mu^*(D_{2n}) \le \mu^*(A)$$

· similarly for odd numbers: $\sum_{n=1}^{\infty} \mu^*(D_{2n+1}) \le \mu^*(A)$

$$\Rightarrow \sum_{n=1}^{\infty} \mu^*(D_n) \le 2\mu^*(A)$$

$$\mu^*(A \cap F^c) = \mu^*\left(E_n \cup \left(\bigcup_{k=n}^{\infty} D_k\right)\right) \le \mu^*(E_n) + \underbrace{\sum_{k=n}^{\infty} \mu^*(D_n)}_{\xrightarrow{\lim_{n\to\infty}} 0}$$

$\forall \epsilon > 0 \ \exists N$ s.t. $\forall n > N$ $\sum_{k=n}^{\infty} \mu^*(D_k) < \epsilon$

$\Rightarrow \mu^*(A \cap F^c) \le \mu^*(E_n) + \epsilon \Rightarrow \mu^*(E_n) \ge \mu^*(A \cap F^c) - \epsilon$

Trivially, $\mu^*(E_n) \le \mu^*(A \cap F^c)$, so $\lim_{n\to\infty} \mu^*(E_n) = \mu^*(A \cap F^c)$

$\therefore$ $\mu^*(A) \ge \mu^*(A \cap F) + \mu^*(E_n) \to \mu^*(A \cap F) + \mu^*(A \cap F^c)$

$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c)$. $\square$

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* Non-Measurable Sets

cosider $[0, 1)$ and operation $x \oplus y = x + y \mod 1$ on $x, y \in [0, 1)$

Lebesgue measure, $m(E) = m(E \oplus x)$ $\forall E \subset [0, 1)$, $x \in [0, 1)$

· equivalence relation: $x \sim y$ if $x - y \in \mathbb{Q}$

$\sim$ splits $[0, 1)$ into cosets E_α

Take 1 element from each coset to create set E ← requires axiom of choice

$\mathbb{Q} \cap [0, 1) = \{r_1, r_2, \ldots\}$ [$\mathbb{Q}$ is countable, so we can do this]

• Claim: ① $\bigcup_{k=1}^{\infty} E \oplus r_k = [0, 1)$

② $(E \oplus r_i) \cap (E \oplus r_j) = \emptyset$ if $i \ne j$

· If E is measurable $\Rightarrow$ $E \oplus r_k$ is measurable

$1 = m([0, 1)) = \sum_{k=1}^{\infty} m(E \oplus r_k) = \sum_{k=1}^{\infty} m(E)$, which is impossible

(1 can't be an infinite sum of constant terms)

Prove claim:

① Take arbitrary $x \in [0, 1)$, then $x \in E_\alpha$ for some E_α

Let $x' \in E$ be representative of E_α

$x - x' = r \in \mathbb{Q}$ and let $r' = r \pmod 1$

Then $x \in E \oplus r'$. $\square$