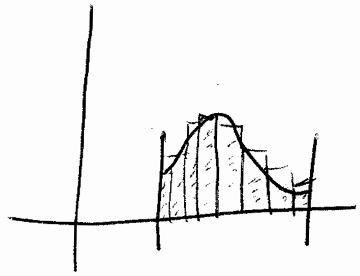
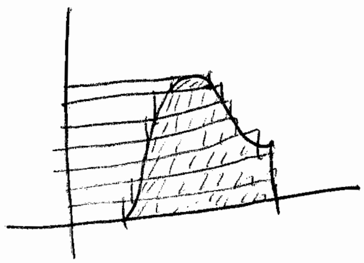
Measurable Functions (Lebesgue Integral)

Riemann Integral:



Lebesgue Integral:



$\overline{\mathbb{R}} = [-\infty, \infty]$, X μ Ω

Def $f: X \to \overline{\mathbb{R}}$ is calle
μ-measurable (Ω-measurable)
if $\forall t \in \mathbb{R}$ $\{x: f(x) > t\} \in \Omega$

Lemma: Let $f: X \to \overline{\mathbb{R}}$, μ, Ω
Then the following are equivalent:

1) $\forall t \in \mathbb{R}$ $\{x: f(x) > t\} \in \Omega$
2) $\geq$
3) $\leq$
4) $<$
5) $f^{-1}(B) \in \Omega$ $\forall$ Borel sets B
and $B = \{+\infty\}$, $B = \{-\infty\}$

Proof of Lemma:

$1 \Leftrightarrow 2$

$\{x : f(x) > t\}^c = \{x : f(x) \le t\}$

$2 \Rightarrow 3$ $\bigcup_{n=1}^{\infty} \{x : f(x) \le t - \frac{1}{n}\}^c = \{x : f(x) \ge t\}$

$3 \Leftrightarrow 4$

$\{x : f(x) \ge t\}^c = \{x : f(x) < t\}$

$4 \Rightarrow 5$

$f^{-1}(\infty) = \left(\bigcap_{n=1}^{\infty} \{x : f(x) < n\}\right)^c$

$f^{-1}(-\infty) = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$

$5 \Rightarrow$ Let $\mathcal{G} \in 2^X$

$\mathcal{G} = \{B : f^{-1}(B) \in \Omega\}$

$\mathcal{G}$ is σ-algebra

because $f^{-1}\left(\bigcup_{k=1}^{\infty} B_k\right) = \bigcup_{k=1}^{\infty} \underbrace{f^{-1}(B_k)}_{\in \Omega}$

We know $(-\infty, t) \in \mathcal{G}$ so $\mathcal{B} \subset \mathcal{G}$ $\square$

<u>Thm:</u> $f, g : X \to \overline{\mathbb{R}}$ are measurable

1) $f+g$ is measurable

2) $|f|$ is measurable

3) f^2 is measurable

4) fg is measurable

If f_n is a sequence of meas functions

5) $\sup(f_n)$ meas

6) $\inf(f_n)$ meas

7) $\limsup(f_n)$ meas

8) $\liminf(f_n)$ meas

9) If $\lim f_n = f \Rightarrow f$ is meas

Pf of Thm:

1: $\{x : f(x) + g(x) > t\} = \bigcup_{q \in \mathbb{Q}} \{x : f(x) > t + q\} \cap \{x : g(x) \geq t - q\}$

2: $\{f : |f(x)| < t\} = \{x : f(x) < t\} \cap \{x : f(x) > -t\}$

3: $\{x : f(x)^2 < t\} = \{x : f(x) < \sqrt{t}\} \cap \{x : f(x) > -\sqrt{t}\}$

3a: cf is measurable because
$\{x : cf(x) > t\} = \{x : f(x) > \frac{t}{c}\}$ for $c > 0$

4: $fg = ((f+g)^2 - f^2 - g^2)/2$

5: Let $f(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$
$\{x : f(x) > t\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > t\}$

6: $-\sup(-f_n(x))$

7: $\limsup f_n(x) = \inf_{n \in \mathbb{N}} \sup_{k \geq n} (f_k(x))$

8: $\liminf f_n(x) = \sup_{n \in \mathbb{N}} \inf_{k \geq n} (f_k(x))$

9: If $\lim f_n = f$ then $\lim f_n = \limsup f_n$

<u>Thm</u>: $f : X \to \overline{\mathbb{R}}$ $\mathcal{O}\!\ell$-measurable
$\phi : \mathrm{Ran}(f) \to \overline{\mathbb{R}}$ continuous
Then $\phi \circ f$ is measurable

<u>Cor</u>: $f : X \to \overline{\mathbb{R}}$ $\mathcal{O}\!\ell$-measurable
$f(x) \neq 0\ \forall x \Rightarrow \frac{1}{f(x)}$ is measurable

Pf: Let $\phi(t) = \frac{1}{t}$ and apply thm.
(If we let $1/0 = \infty$ then $\frac{1}{f(x)}$ is measurable
even if $f(x) = 0$ for some x.)