Dylan Elman

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Lebesgue Integral

Def: Simple function
- measurable function taking finitely many values.

Characteristic function, indicator of a set A

$$\chi_A = 1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Let f be simple

$c_1, c_2, \dots c_n$ - values

$$A_k = f^{-1}(c_k) = \{x : f(x) = c_k\}$$

$$f = \sum_{k=1}^{n} c_k 1_{A_k}$$ - canonical decomposition of f

$$\boxed{\int_A 1 \, d\mu = \mu(A)}$$

$$\int_A 1\, d\mu = \int 1_A \, d\mu$$

if f - simple,

$$\int f \, d\mu = \sum_{k=1}^{n} c_k \mu(A_k)$$

c_k, A_k

Remark Let a simple f admits representation $f = \sum_1^n c_k \mu(A_k)$

where $A_k \cap A_j = \emptyset$ for $j \neq k$

Then $\int f \, d\mu = \sum c_k \mu(A_k)$

(In this representation, some c_k can coincide)

Properties

1) $f \le g \Rightarrow \int f\,d\mu \le \int g\,d\mu$

2) $\int a f\,d\mu = a\int f\,d\mu$

3) $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$

For simple f, g

Proof of 3 :

$$f = \sum_1^n c_k 1_{A_k} \qquad g = \sum_1^m d_j 1_{B_j}$$

$$f+g = \sum_{k=1}^n \sum_{j=1}^m (c_k + d_j)\, 1_{\underbrace{A_k \cap B_j}_{\text{disjoint}}}$$

$$\int (f+g)\,d\mu = \sum_{k,j} (c_k + d_j)\,\mu(A_k \cap B_j)$$

$$\int f\,d\mu = \sum_k \sum_j c_k \mu(A_k \cap B_j)$$

$$f = \sum_{k,j} c_k 1_{A_k \cap B_j}$$

$$\int g\,d\mu = \sum_{k,j} d_j\,\mu(A_k \cap B_j)$$

add these two and get this

Def $f : X \to \overline{\mathbb{R}}$ measurable, $f \ge 0$

$$\int_X f\,d\mu = \sup \{ \int g\,d\mu : g\text{-simple}, 0 \le g \le f \}$$

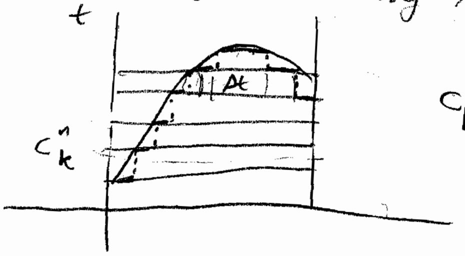
Prop 1 : $f \ge 0$ meas, then

$\exists$ simple f_n $0 \le f_n \le f$ s.t.

$f_n(x) \le f_{n+1}(x)$ $\forall x$, & $\lim f_n(x) = f(x)$ $\forall x$

Prop 2: $\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$

examples of increasing sequence of function



$c_k^n = \frac{k}{2^n}$ $\qquad 1 \le k \le (2^n)^2$

$A_k = \{x : c_k^n < f(x) \le c_{k+1}^n\}$

Prop 3 If $f \geq 0$, meas.

then $\int f \, d\mu = \underbrace{\int_0^\infty \mu\{x : f(x) > t\} \, dt}_{\text{Improper Riemann Integral}}$ $(*)$

$F(t) = \mu\{x : f(x) > t\}$
distribution function of f

$\int f_n \, d\mu$ - Riemann sum for $(*)$

$F(t) \searrow \qquad F(t_2) \le F(t_1)$ if $t_2 > t_1$

F monotone & bdd on $[a,b]$
$\Rightarrow F$ is Riemann integrable

$\lim_{n\to\infty} \int f_n \, d\mu \neq \int \lim_{n\to\infty} f_n$ in general