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①

Prop 2 still not proven...

Monotone Convergence theorem

Thm $f_n \geq 0$, meas, $f_n \nearrow$
$\lim_{n\to\infty} f_n(x) = f(x)$

Then $\lim_{n\to\infty} \int f_n \, d\mu = \int (\lim_{n\to\infty} f_n) \, d\mu = \int f \, d\mu$

Remark f_n - meas, $f = \lim f_n$ meas
$\int f d\mu = \sup \{ \int g \, d\mu : g\text{-simple}, g \leq f \}$

L Let ϕ-simple function
$\nu(E) = \int_E \phi \, d\mu = \int 1_E \phi \, d\mu$ Then ν-measure

Pf Trivial if $\phi = 1_A$
In this case $\nu(E) = \mu(A \cap E)$ and we can easily check count. add.

true for $c \cdot 1_A$ $c > 0$
$\Rightarrow$ true for $\sum_1^n c_k 1_{A_k}$ $c_k > 0$

Pf of MCT

Obs $0 \leq f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$
ϕ-simple $\phi \leq f \Rightarrow \phi \leq g$
$\Rightarrow \int \phi d\mu \leq \int g d\mu$
$\sup \{ \int \phi d\mu : 0 \leq \phi \leq f \} \leq \int g d\mu$
$\quad\quad \shortparallel$
$\quad \int f d\mu$

So $\int f_n d\mu \leq \int f d\mu$

so $\lim_{n\to\infty} \int f_n d\mu \leq \int f d\mu$

$(\geq)$ WTP (want to prove)

Take simple $0 \leq \varphi \leq f$

$\alpha \in (0,1)$ $\alpha \approx 1$

$E_n = \{x : f_n(x) \geq \alpha \varphi(x)\}$

meas.

$E_n \subset E_{n+1}$

<u>Claim</u> $\bigcup_1^\infty E_n = X$

Indeed, if $\exists\, x \notin \bigcup_1^\infty E_n \Rightarrow f_n(x) < \alpha \varphi(x) \ \forall n$

only happens if $\varphi(x) > 0$

$\lim f_n(x) \leq \alpha \varphi(x) \leq \alpha f(x) < f(x)$

$\overset{\shortparallel}{f(x)}$ $\Rightarrow\Leftarrow$

$\nu(E) = \int_E \varphi \, d\mu$

$$\int f_n \, d\mu \geq \underbrace{\int_{E_n} f_n \, d\mu}_{\int f_n 1_{E_n} d\mu} \geq \alpha \int_{E_n} \varphi \, d\mu = \nu(E_n)$$

$n \to \infty$

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \alpha \int \varphi \, d\mu = \nu\left(\bigcup_1^\infty E_n\right)$$

follows from countable additivity of ν

Take sup over simple φ, $0 \leq \varphi \leq f$

$\lim_{n\to\infty} \int f_n \, d\mu \geq \alpha \int f \, d\mu \quad \forall \alpha \in (0,1)$

$\lim \int f_n \, d\mu \geq \int f \, d\mu$

$\boxed{\text{Prop 2 is proved now}}$

Corollary

Let $\varphi \geq 0$ measurable,

$\nu(E) = \int_E \varphi \, d\mu \quad \forall$ meas. sets E

Then ν is a measure.

Pf $\nu(E) \geq 0$, so WTS count. [redacted] add.

$\{E_n\}_1^\infty$, $E_n \cap E_k = \emptyset$ if $n \neq k$

$f_n = \varphi 1_{\bigcup_1^n E_k}$, $\quad f = \varphi 1_E \quad E = \bigcup_1^\infty E_k \Rightarrow f_n \nearrow f$

$$\sum_{k=1}^n \nu(E_k) = \int f_n \, d\mu \xrightarrow[n\to\infty]{} \int f \, d\mu = \int_E \varphi \, d\mu = \nu(E)$$

$\downarrow n \to \infty$

$$\sum_{k=1}^\infty \nu(E_k) = \nu(E)$$

$f_n = n \, 1_{(0, \frac{1}{n})}$

$\lim f_n(x) = 0$

$\int_{\mathbb{R}} f_n(x) \, dx = 1 \quad \longleftarrow$ example where $\lim\limits_{n\to\infty} \int f_n \, d\mu \neq \int \lim\limits_{n\to\infty} f_n \, d\mu$

Cor $f, g \geq 0$ meas

then $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

Pf $0 \leq f_n \nearrow f$, $g_n \nearrow g$ (Prop 1)

Then $\underbrace{f_n + g_n}_{\text{simple}} \nearrow f+g$

$$\lim_{n\to\infty} \int (f_n + g_n) \, d\mu = \lim_{n\to\infty} \left(\int f_n \, d\mu + \int g_n \, d\mu \right) = \lim_{n\to\infty} \int f_n \, d\mu + \lim_{n\to\infty} \int g_n \, d\mu$$

$$\| \qquad\qquad\qquad\qquad = \int f \, d\mu + \int g \, d\mu$$

$\int (f+g) \, d\mu$

Prop 2 or MCT (of which Prop 2 is a particular case)

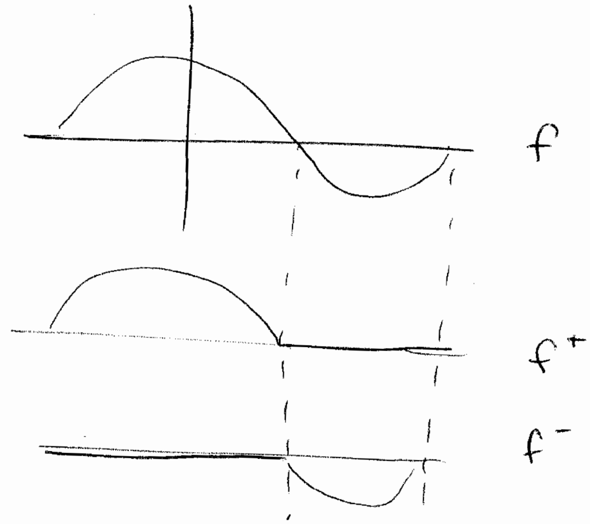
Integral of general functions

f meas.

$f_+(x) = \max\{f(x), 0\}$

$f_-(x) = \max\{-f(x), 0\}$

$f = f_+ - f_-$



Define $\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu$

$\infty - \text{finite} = \infty$

$\text{finite} - \infty = -\infty$

$\infty - \infty$ not defined

Def meas. f is called summable ($f \in L^1$) iff $\int f_+ \, d\mu < \infty$, $\int f_- \, d\mu < \infty$

iff $\int |f| \, d\mu < \infty$

f-meas. $\Rightarrow$ f_+ meas because $f_+ = \dfrac{f + |f|}{2}$

Prop $f, g \in L^1(\mu)$

$\Rightarrow \int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

not trivial because $(f+g)_+ \neq f_+ + g_+$