#1 $(X+\Delta X)^3 =$

$$= X^3 + \Delta X \cdot X^2 + X \Delta X \cdot X + X^2 \Delta X$$

$+\{$terms with more than 1 $\Delta X\}$

Each such term can be estimated by $C \cdot \|\Delta X\|^2$ as $\|\Delta X\| \to 0$

so each term $= o(\Delta X)$

Therefore $dX^3$ is a linear transformation

$$\Delta X \mapsto \Delta X\, X^2 + X \Delta X\, X + X^2 \Delta X$$

# 2. $(X + \Delta X)^{-1} =$

$$= \left[ X (I + X^{-1} \Delta X) \right]^{-1}$$

$$= \left[ (I + X^{-1} \Delta X)^{-1} X^{-1} \right]$$

$$= \left[ \sum_{k=0}^{\infty} (X^{-1} \Delta X)^k (-1)^k \right] X^{-1}$$

series converges if $\| X^{-1} \Delta X \| < 1$

$$= X^{-1} - X^{-1} \Delta X \cdot X^{-1} + \sum_{k=2}^{\infty} (X^{-1} \Delta X) (-1)^k X^{-1}$$

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$$\left\| \sum_{k=2}^{\infty} (-1)^k (X^{-1} \Delta X)^k X^{-1} \right\| \leq$$

$$\sum_{k=2}^{\infty} \| X^{-1} \|^{k+1} \| \Delta X \|^k = \frac{\| X^{-1} \|^3 \| \Delta X \|^2}{1 - \| X^{-1} \| \cdot \| \Delta X \|} = o(\Delta X)$$

So $d(X^{-1})$ is a linear

map $\Delta X \longmapsto -X^{-1} \Delta X X^{-1}$

\# 3 $\det(X+\Delta X) =$

$$\det\left(X\left(I + X^{-1}\Delta X\right)\right)$$

$$= \det X \cdot \det\left(I + X^{-1}\Delta X\right)$$

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$\det(A) =$ Product of eigenvalues

$\operatorname{trace}(A) =$ sum of eigenvalues

(both counting multiplicities)

Let $\lambda_1, \lambda_2 \ldots \lambda_n$ be eigenvalues of $X^{-1}\Delta X$.

Note that $|\lambda_k| \le \|X^{-1}\Delta X\| \le \|X^{-1}\| \cdot \|\Delta X$

$$\det\left(I + X^{-1}\Delta X\right) = \prod_{k=1}^{n}(1+\lambda_k)$$

$$= 1 + \sum_{k=1}^{n}\lambda_k$$

$+$ (sum of products with more than 1 $\lambda_k$)

The latter sum can be estimated by $C\,\|\Delta X\|^2 = o(\Delta X)$,

So

$$(*) \quad \det(I + X^{-1}\Delta X) = 1 + \operatorname{trace}(X^{-1}\Delta X) + o(\Delta X)$$

Therefore

$$\det(X + \Delta X) = \det X + \det X \operatorname{trace}(X^{-1}\Delta X) + o(\Delta X)$$

so $(\det X)'$ is a linear map

$$\Delta X \mapsto \det X \cdot \operatorname{trace}(X^{-1}\Delta X)$$

Altermative way to get $(*)$:

Compute derivative $f'(I)$ for $f(X) = \det(X)$.

Partials:

$$\left.\frac{\partial f}{\partial x_{ij}}\right|_{X=I} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

($x_{ij}$ - entries of $X$)

Then, for small $H \in M_{n\times n}$

$$\det(I+H) = \sum_{j,k=n}^{n} \frac{\partial f}{\partial x_{jk}} H_{jk} + o(H)$$

$$= \sum_{j=1}^{n} H_{jj} + o(H)$$

$$= \operatorname{trace} H + o(H)$$

To get (*) we only need to show that $o(H) = o(\Delta X)$ for $\Delta H = X^{-1}\Delta X$

To do this let us compute

$$\lim_{\|\Delta X\| \to 0} \frac{\|r(X^{-1}\Delta X)\|}{\|\Delta X\|}$$

$$= \lim_{\|\Delta X\| \to 0} \underbrace{\frac{\|r(X^{-1}\Delta X)\|}{\|X^{-1}\Delta X\|}}_{\to 0 \text{ because } r(H) = o(H)} \; \underbrace{\frac{\|X^{-1}\Delta X\|}{\|\Delta X\|}}_{\leq \|X^{-1}\|}$$

$= 0$ (because $o(1) \cdot O(1) = o(1)$)