Homework assignment, September 17, 2007.

1. For sets $A, B \subset \mathbb{R}$, the sum $A + B$ is defined as $A + B := \{a + b : a \in A, b \in B\}$. Prove that

$$\sup(A + B) = \sup A + \sup B.$$

2. For a function $f : X \to \mathbb{R}$ the notation $\sup_{x\in A} f(x)$ is used for the supremum of the set $\{f(x) : x \in A\}$. Give an example showing that

$$\sup_{x\in A}(f(x) + g(x)) < \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

Note that the inequality $\leq$ always holds.

3. For the sequence $a_n = (-1)^n + 1/n$ compute $\limsup a_n$ and $\liminf a_n$. Give all the details.

4. Prove that if $\lim a_n = A$ then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = A.$$

Show that the converse is not true.

This problem is probably the hardest in the assignment. If you can do it, you are pretty comfortable working with ε–δ.

5. Show that if

$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty$$

then the sequence a_n converges, but the converse is not true.