Homework assignment, Oct. 5, 2007.

1. Give an example of a subset (of a complete metric space) which is

   a) bounded but not compact;

   b) closed, but not compact;

   c) closed and bounded, but not compact.

2. For sets A and B in a metric space, the distance $\mathrm{dist}(A, B)$ between A and B is defined as

$$\inf\{\rho(x, y) : x \in A, y \in B\}.$$

   a) Show that if A and B are compact, then then the distance between A and B is attained, i.e. there exist $a \in A$, $b \in B$ such that $\rho(a, b) = \mathrm{dist}(A, B)$.

   b) Show that the above statement is not true if we only assume that A and B are closed

   c) Show that if $A, B \subset \mathbb{R}^d$, A is compact, B is closed, then the distance between A and B is attained.

3. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Assume that f is homogeneous of degree r, $r > 0$, meaning that

$$f(t\mathbf{x}) = t^r f(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^d, \quad \forall t \in [0, \infty).$$

Show that if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, then

$$f(\mathbf{x}) \geq c|\mathbf{x}|^r \qquad \forall \mathbf{x} \in \mathbb{R}^d$$

for some $c > 0$. Here $|\mathbf{x}|$ is the Euclidean length of the vector $\mathbf{x}$. **Hint:** Use compactness.